Krishna's

B.Sc. Objective

1000+
MCQs

Differential Equations and Integral Transforms

(For B.Sc. II year Students of all Colleges affiliated to Universities in U.P.)

As Per U.P. Unified Syllabus

By

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Syllabus

Differential Equations and Integral Transforms

B.Sc. II Year; II Paper

As per U.P. UNIFIED Syllabus (w.e.f. 2012-13)

M.M. : 33/65

Differential Equations

Unit 1

Formation of a differential equation (D.E.), Degree, order and solution of a D.E., Equations of first order and first degree : Separation of variables method, Solution of homogeneous equations, linear equations and exact equations, Linear differential equations with constant coefficients, Homogeneous linear differential equations.

Unit 2

Differential equations of the first order but not of the first degree, Clairaut's equations and singular solutions, Orthogonal trajectories, Simultaneous linear differential equations with constant coefficients, Linear differential equations of the second order (including the method of variation of parameters).

Unit 3

Series solutions of second order differential equations, Legendre and Bessel functions (P_n and J_n only) and their properties.

Order, degree and formation of partial differential equations, Partial differential equations of the first order, Lagrange's equations, Charpit's general method, Linear partial differential equations with constant coefficients.

Unit 4 (i)

Partial differential equations of the second order, Monge's method.

Integral Transforms

Unit 4 (ii)

The concept of transform, Integral transforms and kernel, Linearity property of transforms, Laplace transform, Inverse Laplace transform, Convolution theorem, Applications of Laplace transform to solve ordinary differential equations.

Unit 5

Fourier transforms (finite and infinite), Fourier integral, Applications of Fourier transform to boundary value problems, Fourier series.

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Chapter 2: Linear Differential Equations with Constant Coefficients....(12-20)
Chapter 3: Homogeneous Linear Differential Equations....(21-26)

Unit-II

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Chapter 5: Orthogonal Trajectories....(35-39)
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B.Sc. Objective Mathematics 2.2

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Differential Equations and Integral Transforms

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Chapter -13 : Series Solutions of Differential Equations
Chapter -14 : The Laplace Transform

Contd...

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Unit-IV

Chapter -15 : The Inverse Laplace Transform
Chapter -16 : Applications of Laplace Transform

Unit-V

Chapter -17 : Fourier Transforms
Chapter -18 : Finite Fourier Transform
Chapter -19 : Applications of Fourier Transforms in Initial and Boundary Value Problems
Chapter -20 : Fourier Series

CHAPTER

1 Differential Equations of First Order and First Degree

DEFINITIONS

1. An equation which involves differential coefficients or derivatives is called a differential equation.
2. A differential equation is said to be partial if there are two or more independent variables.
3. The order of a differential equation is the order of the highest derivative occuring in the equation.
4. The degree of a differential equation is the highest power of the highest order derivative present in it when the derivative occurs in the form of polynomials *i.e.* made free from radicals and fractions so far as derivatives are concerned.
5. A differential equation is said to be linear if the dependent variable and all its derivatives occur in the first degree and they do not occur together with multiple operation, otherwise called non-linear.
6. The function $y = f(x)$ is said to be the solution of a differential equation if, when putting in the equation it reduces the equation to an identity.
7. If in a differential equation, the independent arbitrary constants are equal to the order of the differential equation then the solution is called the general solution.
8. A solution obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution or integral.
9. The solution of an n^{th} order differential equation may contain n arbitrary constants.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

1. The differential equation of first order and of first degree is represented by

$$Mdx + Ndy = 0$$

where M and N are some functions of x and y or are constants.

2. If the differential equation can be written in the form $f(x)\,dx = \phi(y)dy$

where $f(x)$ is a function of x only and $\phi(y)$ is a function of y only then we get the solution by integrating both sides and we say that the variables are separable.

3. A differential equation of the form

$\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$ where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equations in such cases we put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and then solve it by variables separable method.

4. The differential equation of the form

$\dfrac{dy}{dx} = \dfrac{ax + by + c}{Ax + By + c}$ can be reduced to the homogeneous form by substitution

$x = X + h$, $y = Y + k$ and so $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

The given differential equation reduced to

$\dfrac{dY}{dX} = \dfrac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + C}$, choose h and k such that

$$ah + bk + c = 0 \text{ and } Ah + Bk + C = 0$$

Then given equation becomes $\dfrac{dY}{dX} = \dfrac{aX + bY}{AX + BY}$

and solve by substituting $Y = VX$.

This is called equations reducible to homogeneous form.

5. An equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only is called a linear differential equation of the first order. In such case, multiply both sides by $e^{\int Pdx}$ *i.e.*

$$e^{\int Pdx}.\left(\frac{dy}{dx} + Py\right) = Q.e^{\int Pdx}$$

i.e. $$\frac{d}{dx}\left(y \,.\, e^{\int Pdx}\right) = Q \cdot e^{\int Pdx}$$

Integrating both sides, we get

$y\, e^{\int Pdx} = \int Q.e^{\int Pdx} + C$ which is a solution.

6. The differential equation $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x alone is called Bernoulli's equation.

On dividing both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$$

Put $y^{-n+1} = V$ so $(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dV}{dx}$ it reduces to $\frac{dV}{dx} + (1-n)\, P.V = (1-n)Q$ which is linear.

It is called equations reducible to the linear from.

7. An exact differential equation is formed by directly differentiating its primitive *i.e.* solution without any other process.

The necessary and sufficient condition for the arbitrary differential equation $Mdx + Ndy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the differential equation $Mdx + Ndy = 0$ is exact then its solution is

$$\underset{y\text{ as constant}}{\int Mdx} + \underset{\substack{\text{Only those}\\ \text{terms in}\\ N\text{ which do}\\ \text{not contain }x}}{\int Ndy} = C$$

EQUATION REDUCIBLE TO THE EXACT EQUATIONS (INTEGRATING FACTORS)

1. A differential equation which is not exact can sometimes be made exact by multiplying by some suitable function of x and y then such function is called an integrating factor. This integrating factor can by formed by inspection.

2. If the differential equation $Mdx + Ndy = 0$ is of the form

   $[f(x, y)]\, ydx + [\phi\,(x, y)]\, x\; dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor.

3. If the differential equation $Mdx + Ndy = 0$ is homogeneous equation and $Mx + Ny \neq 0$ then $\frac{1}{Mx + Ny}$ is an integrating factor.

4. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ is a function of x alone then the integrating factor for $Mdx + Ndy = 0$ is $e^{\int f(x)dx}$.

5. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ is a function of y alone, then the integrating factor for $Mdx + Ndy = 0$ is $e^{\int f(y)\, dy}$.

6. If the differential equation is

   $x^a y^b\,(mydx + nxdy) + x^c y^d\,(pydx + qxdy) = 0$

   Where a, b, m, n c, d, p, q are constants, then its integrating factor is $x^h y^k$, where h and k can be obtained by multiplying the given equation by $x^h y^k$ and equation must become exact.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. $y = Ae^x + Be^{-x}$, is the general solution of the differential equation **(K.U. 2015)**
(a) $\frac{d^2y}{dx^2} = -y$ (b) $\frac{d^2y}{dx^2} + y + 1 = 0$
(c) $\frac{d^2y}{dx^2} - y = 1$ (d) $\frac{d^2y}{dx^2} = y$

2. The differential equation $\frac{d^2y}{dx^2} + \sin y = \sin x$ is **(K.U. 2014)**
(a) Linear of order 2 (b) Non linear of order 2
(c) Linear of degree 2 (d) Non linear of degree 2

3. The order and degree of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0$ are
(a) 2 and 3 (b) 3 and 2 (c) 2 and 1 (d) 3 and 4

4. The differential equation corresponding to the equation $y = A\cos x + B\sin x$ is
(a) $y'' + y' + y = 0$ (b) $y'' + 2y' + y = 0$
(c) $y'' + y = 0$ (d) $y' + y = 0$

5. The solution of $(1 + x)\,y dx + (1 - y)\,x\,dy = 0$ is
(a) $x + y = ce^{x - y}$ (b) $x - y = ce^{x + y}$
(c) $xy = ce^{x - y}$ (d) $xy = ce^{y - x}$

6. If for differential equation $\frac{dy}{dx} = e^{x + y}$, $y(1) = 1$ then $y(-1)$ is equal to
(a) 1 (b) – 1 (c) 0 (d) 2

7. The solution of $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is
(a) $x = ke^{y/x}$ (b) $y = ke^{x/y}$
(c) $y = ke^{y/x}$ (d) $y = ke^{x}$

8. If the function 1 and $-e^{-3x}$ are the fundamental solutions of the equation $y'' + py' - 3y = 0$ then, p is
(a) 0 (b) 1 (c) –1 (d) 3

9. The degree of the differential equation $y'' + \sqrt{1 + y'} = 0$ is
(a) 1 (b) 2 (c) 3 (d) 0

10. The integrating factor of $\frac{dy}{dx} + 2xy = e^{-x^2}$ is
(a) e^{x} (b) e^{x^2} (c) e^{x^3} (d) xe^{x}

11. The solution of $\frac{dy}{dx} + y = e^{-x}$ is
(a) $ye^{x} = x + c$ (b) $xe^{y} = x + c$
(c) $xe^{y} = y + c$ (d) $ye^{x} = y + c$

12. The integrating factor of $(x + 2y^3)\frac{dy}{dx} = y$ is
(a) $\frac{1}{x}$ (b) y (c) x (d) $\frac{1}{y}$

13. $y = Ax - A^2$, is the general solution of the differential equation
(a) $(y')^2 - xy' + y = 0$ (b) $y'' + y' = 0$
(c) $(y')^2 + y = 0$ (d) $y'' - y' = A$

14. The differential equation whose solution is $y = a\cos(x + 3)$, is
(a) $y' + \tan(x + 3) = 0$ (b) $y'' + \tan(x + 3) = 0$
(c) $y' - \tan(x + 3) = 0$ (d) $y'' - \tan(x + 3) = 0$

15. The solution of $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$ is
(a) $\cot x \tan y = c$ (b) $\tan x \cot y = c$
(c) $\cot x \cot y = c$ (d) $\tan x \tan y = c$

16. The solution of $\cos(x + y)dx = dy$ is
(a) $\tan(x + y) = c$ (b) $\cos(x + y) = c$
(c) $\sin\left(\frac{x + y}{2}\right) = x + c$ (d) $\tan\left(\frac{x + y}{2}\right) = x + c$

17. The solution of $x\frac{dy}{dx} + \frac{y^2}{x} = y$ is
(a) $cx = e^{x/y}$ (b) $cy = e^{x/y}$
(c) $cx = e^{x-y}$ (d) $cy = e^{y-x}$

18. The differential equation $yx' = y - 1$, $y(0) = 1$ has
(a) Infinite number of solutions
(b) Unique solution
(c) No solution (d) Many solutions

19. The integrating factor of $\cos^2 x\frac{dy}{dx} + y = \tan x$ is
(a) $\tan x$ (b) $\cot x$ (c) $e^{\cot x}$ (d) $e^{\tan x}$

20. The solution of $(y\cos x + 1)dx + \sin x\,dy = 0$ is
(a) $y + x \sin x = c$ (b) $y + x\cos x = c$
(c) $x + y\sin x = c$ (d) $x + y\cos x = c$

21. The integrating factor of $x\frac{dy}{dx} + y = y^2\log x$ is
(a) $\frac{1}{x^2}$ (b) $\frac{1}{x}$ (c) $\frac{1}{y}$ (d) $\frac{1}{y^2}$

22. The differential equation $Mdx + Ndy = 0$ is exact when

(a) $Mx = Ny$ (b) $My = Nx$

(c) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ (d) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

23. The solution of $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)\,dy = 0$ is

(a) $x + y\,e^{x/y} = c$ (b) $y + e^{x/y} = c$

(c) $x + e^{x/y} = c$ (d) None of these

24. If $x^h y^k$ be the integrating factor of $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ then h and k are

(a) $-5/2, 1/2$ (b) $5/2, 1/2$

(c) $5/2, -1/2$ (d) $-5/2, -1/2$

25. The integrating factor of $(x^2y^2 + xy + 1)y dx + (x^2y^2 - xy + 1)x\,dy = 0$ is

(a) $\frac{1}{2xy}$ (b) $\frac{1}{2x^2y}$

(c) $\frac{1}{2xy^2}$ (d) $\frac{1}{2x^2y^2}$

26. If the polar subtangent is constant then the curve is

(a) $r(\theta - c) = a$ (b) $r\theta = a$

(c) $r(\theta + c) + a = 0$ (d) $r + \theta = a$

27. If the equation $y \sin x\, dx - \sin x\, dy = 0$ is exact then x is equal to

(a) 0 (b) $\pi/2$ (c) $\pi/3$ (d) $\pi/4$

28. The solution of $\frac{dy}{dx} + \frac{y}{x} = x^2$ if $y(1) = 1$ is

(a) $xy = \frac{x^2}{4} + c$ (b) $xy = \frac{x^4}{4} + \frac{3}{4}$

(c) $xy = \frac{x^2}{4} + \frac{3}{4}$ (d) $xy = \frac{x^2}{4} + \frac{3}{4}$

29. The differential equation

$$xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$$ is

(a) Homogeneous (b) Of order 2

(c) Linear (d) Exact

30. The solution of $(x^2 + y^2)\,dx - 2xy\,dy = 0$ is

(a) $x - y = cx$ (b) $x^2 + y^2 = cx$

(c) $x^2 - y^2 = cx$ (d) $x + y = cx$

31. The solution of $x\frac{dy}{dx} = y(\log y - \log x + 1)$ is

(a) $y = xe^{cx}$ (b) $x = ye^{cx}$

(c) $y = x^2 + c$ (d) $x = y + c$

32. In homogeneous equation the degree of each term is

(a) Different (b) One

(c) Two (d) Same

33. If the equation $(x^2 + ay + x)dx + (y^2 + x - y)\,dy = 0$ is exact then a is

(a) 0 (b) 1 (c) 2 (d) Not exist

34. If the subnormal is constant then the curve is

(a) Circle (b) Ellipse

(c) Hyperbola (d) Parabola

35. The differential equation $\frac{dx}{dy} + \frac{1}{y^2}x = \frac{\tan^{-1}y}{1+y^2}$ is linear only when dependent variable is/are

(a) x (b) y

(c) Either x or y (d) Neither x nor y

36. The differential equation $(y^2e^x + 2xy)dx - x^2dy = 0$ is

(a) Exact (b) Not exact

(c) Linear (d) Homogeneous

37. The number of arbitrary constants in the solution of differential equation of order 2 is

(a) 1 (b) 2 (c) 3 (d) 4

38. The order of the differential equation

$$\left[4 + \left(\frac{d^3y}{dx^3}\right)^2\right]^{2/3} = \frac{d^2y}{dx^2} + 1$$ is

(a) 1 (b) 2 (c) 3 (d) 4

39. The degree of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2 = \left(\frac{d^2y}{dx^2}\right)^{3/2}$$ is

(a) 1 (b) 3 (c) 2 (d) 3/2

40. The solution of $(xy + x)dy - (xy + y)dx = 0$ is

(a) $x = cye^{y-x}$ (b) $y = cxe^{y-x}$

(c) $xy = ce^{y-x}$ (d) $xy = ce^{x-y}$

41. The solution of $\frac{dy}{dx} = 1 + \tan(y - x)$ is

(a) $\sin(y + x) = ce^x$ (b) $\cos(y - x) = ce^x$

(c) $\tan(y - x) = ce^x$ (d) $\sin(y - x) = ce^x$

42. The solution of $\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$ is

(a) $\tan y/x = cx$ (b) $\sin y/x = cx$

(c) $\cos y/x = cx$ (d) $\cot y/x = cx$

43. The solution of

$x \cos x\, dy + (xy \sin x + y \cos x - 1)\ dx = 0$

(a) $xy = c \sin x \cdot \cos x$

(b) $x - y = \sin x + c \cos x$

(c) $x + y = \sin x + c \cos x$

(d) $xy = \sin x + c \cos x$

44. The solution of $\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$ is

(a) $\sin x \sec y = x + c$ (b) $\sin y \sec x = x + c$

(c) $\sin x \sec x = y + c$ (d) $\sin x \cos y = x + c$

45. If $x^h y^k$ be the integrating factor of $(3y - 2xy^3)dx + (4x - 3x^2y^2)\ dy = 0$ then h and k is equal to

(a) 2, 3 (b) 3, 2 (c) 2, 4 (d) 3, 4

46. The integrating factor of $\frac{dy}{dx} + Py = Q$ is

(a) $e^{\int Qdy}$ (b) $e^{\int Pdy}$ (c) $e^{\int Qdx}$ (d) $e^{\int Pdx}$

47. The integrating factor of $(y + 2x^3)\ dx - x\ dy = 0$ is

(a) $1/x^2$ (b) x (c) e^x (d) e^{-x}

48. The solution of $y' + y \tan x = \cos x$, $y(0) = 0$ is

(a) $x \sin x$ (b) $x \cos x$ (c) $\sin x$ (d) $\cos x$

49. In solving $\frac{dy}{dx} = \frac{x + y + 2}{x - y + 3}$ the substitution are

(a) $x = X + h,\ y = Y - k$

(b) $x = X^2 + h,\ y = Y^2 + k$

(c) $x = X + h,\ y = Y + k$

(d) $x = X + 1,\ y = Y + 1$

50. The differential equation $\left(\frac{dy}{dx}\right)^{3/2} + y = x^{2/3}$ is

(a) Linear (b) Non linear of order 1

(c) Non linear of order 3/2

(d) Linear of order 1

51. The differential equation $\frac{dy}{dx} + Py = Qy^n$ can be reduced to linear form by substituting

(a) $\frac{1}{y^{n-1}} = V$ (b) $\frac{1}{y^n} = V$

(c) $\frac{1}{y^{n+1}} = V$ (d) $\frac{1}{y} = V$

52. The differential equation $yx' = y - 1$ with $x(1) = 0$ has

(a) Unique solution

(b) Infinite number of solutions

(c) No solution (d) Only two solutions

53. If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$ then its integrating factor is

(a) $Mx + Ny$ (b) $Mx - Ny$

(c) $\frac{1}{Mx + Ny}$ (d) $\frac{1}{Mx - Ny}$

54. If $f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone in the differential equation $Mdx + Ndy = 0$ then Integrating factor is

(a) $e^{\int ydx}$ (b) $e^{\int xdy}$ (c) $e^{\int f(y)\,dx}$ (d) $e^{\int f(x)dx}$

55. The solution of $\cos(x + y)dx = dy$ is

(a) $\tan\left(\frac{x + y}{2}\right) = y + c$ (b) $\tan\left(\frac{x + y}{2}\right) = x + c$

(c) $\tan\left(\frac{x + y}{2}\right) = x + y + c$

(d) None of these

56. The differential equation corresponding to $y = Ae^{2x} + Be^{-3x} + Ce^x$ is

(a) $y''' - 5y' + 3y = 0$ (b) $y''' - 7y' + 6y = 0$

(c) $y''' - 7y' + y = 0$ (d) $y''' + 7y' - 3y = 0$

57. The differential equation corresponding to $y = e^x\ (A \cos x + B \sin x)$ is

(a) $y'' - 2y' + 2y = 0$ (b) $y'' - 2y' + y = 0$

(c) $y'' - y' + 2y = 0$ (d) $y'' - 2y' + 3y = 0$

58. The solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$ is

(a) $y = Ax + B$ (b) $y = A/x + B$

(c) $x = A/y + B$ (d) $x = Ay + B$

59. The solution of $\frac{dy}{dx} = \frac{-2xy}{x^2 + 1}$ at $y(1) = 2$ is

(a) $y(x^2 + 1) = 1$ (b) $x(y^2 + 1) = 4$

(c) $y(x^2 + 1) = 4$ (d) $x(y^2 + 1) = 1$

60. The solution of $x\frac{dy}{dx} + \frac{y^2}{x} = y$ at $y(1) = 1$ is

(a) $ex = e^{x/y}$ (b) $x = e^{x/y}$

(c) $ey = e^{x/y}$ (d) $y = e^{x/y}$

61. The solution of $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ at $y(1) = \pi/2$ is

(a) $\cos\left(\frac{y}{x}\right) = x$ (b) $\tan\left(\frac{y}{x}\right) = x$

(c) $\sin\left(\frac{y}{x}\right) = x$ (d) $\cos\left(\frac{x}{y}\right) = x$

62. The differential equation $(3a^2x^2 + by\cos x)\,dx + (2\sin x - 4ay')dy = 0$ is exact when

(a) $a = 2, b = 3$ (b) $a = 3, b = 2$

(c) $a = 3, b = 3$ (d) $a = 2, b = 2$

63. The solution of the differential equation $\sin x \dfrac{dy}{dx} = y \log y$ with $y \dfrac{\pi}{2} = e$ is

(a) $e^{\cot x/2}$ (b) $e^{\sin x/2}$ (c) $e^{\cos x/2}$ (d) $e^{\tan x/2}$

64. The solution of $\dfrac{dy}{dx} = 2\sqrt{y}$ with $y(0) = 0$ is

(a) No solution (b) Unique solution

(c) Infinite solution (d) Finite solution

65. The differential equation of all circles passing through the origin and having their cenres on the y-axis is

(a) $(x^2 - y^2)\, y' + xy = 0$

(b) $(y^2 - x^2)\, y' + y = 0$

(c) $(y^2 - x^2)\, y' + 2xy = 0$

(d) $y'' + xy' + 2xy = 0$

66. The solution of $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$ is

(a) $y = c(1 - ay)(x + a)$ (b) $y = ay\,(x + a)$

(c) $y = c(1 + ay)\,x$ (d) $y = c(1 + ay)(x - a)$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(b)	**3.**	(a)	**4.**	(c)	**5.**	(d)	**6.**	(b)	**7.**	(c)	**8.**	(a)	**9.**	(b)	**10.**	(b)
11.	(a)	**12.**	(d)	**13.**	(a)	**14.**	(a)	**15.**	(d)	**16.**	(d)	**17.**	(a)	**18.**	(b)	**19.**	(d)	**20.**	(c)
21.	(b)	**22.**	(d)	**23.**	(a)	**24.**	(d)	**25.**	(d)	**26.**	(c)	**27.**	(d)	**28.**	(b)	**29.**	(d)	**30.**	(c)
31.	(a)	**32.**	(d)	**33.**	(b)	**34.**	(d)	**35.**	(a)	**36.**	(b)	**37.**	(b)	**38.**	(c)	**39.**	(b)	**40.**	(a)
41.	(d)	**42.**	(b)	**43.**	(d)	**44.**	(b)	**45.**	(a)	**46.**	(d)	**47.**	(a)	**48.**	(b)	**49.**	(c)	**50.**	(b)
51.	(a)	**52.**	(a)	**53.**	(c)	**54.**	(d)	**55.**	(b)	**56.**	(b)	**57.**	(a)	**58.**	(b)	**59.**	(c)	**60.**	(a)
61.	(c)	**62.**	(b)	**63.**	(d)	**64.**	(b)	**65.**	(c)	**66.**	(a)								

HINTS AND SOLUTIONS

1. $y = Ae^x + Be^{-x}$ differentiating two times

$$\frac{dy}{dx} = Ae^x - Be^{-x},\ \frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

so, $\dfrac{d^2y}{dx^2} = y$

5. $(1 + x)\,ydx + (1 - y)\,xdy = 0$

$$\left(\frac{1+x}{x}\right)dx + \left(\frac{1-y}{y}\right)dy = 0$$

$$\Rightarrow \left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} - 1\right)dy = 0$$

integrating it we get $\log x + x + \log y - y = \log c$

i.e. $\log\left(\dfrac{xy}{c}\right) = y - x$ i.e. $xy = ce^{y-x}$

7. $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$ put $y = vx$ i.e. $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{(v-1)x^2} = \frac{v^2}{v-1}$$

so $x\dfrac{dv}{dx} = \dfrac{v^2}{v-1} - v = \dfrac{v}{v-1}$

i.e. $\left(\dfrac{v-1}{v}\right)dv = \dfrac{dx}{x} \Rightarrow \left(1 - \dfrac{1}{v}\right)dv - \dfrac{dx}{x} = 0$

Integrating $v - \log v - \log x = \log c$

$\Rightarrow \quad \log c\left(\dfrac{xv}{c}\right) = v$

i.e. $\log cy = y/x \Rightarrow cy = e^{y/x} \Rightarrow y = k\,e^{y/x}$

10. The integrating factor is $e^{\int 2xdx} = e^{x^2}$

11. $(D+1)y = e^{-x}$ i.e. $\dfrac{dy}{dx} + y = e^{-x}$ I.F. is $e^{\int 1\,dx} = e^{x}$

So solution is $y\,e^{x} = \int e^{x}, e^{-x} dx + C = x + C$

13. $y = Ax - A^2,\ y' = A$

So $(y')^2 - xy' + y = A^2 - Ax + Ax - A^2 = 0$

15. $\dfrac{\sec^2 x}{\tan x}dx + \dfrac{\sec^2 y}{\tan y}dy = 0$ integrating, we get

$\log \tan x + \log \tan y = \log c$

$\log(\tan x \tan y) = \log c \Rightarrow \tan x \tan y = c$

16. Put $x + y = v,\ 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ put it in

$\cos(x+y)dy = dx$

We get $\cos v\left(\dfrac{dv}{dx} - 1\right) = 1$ *i.e.* $\dfrac{\cos v\,dv}{1+\cos v} = dx$

i.e. $\left(1 - \dfrac{1}{1+\cos v}\right)dv = dx$

or $\left(1 - \dfrac{1}{2}\sec^2\dfrac{v}{2}\right)dv = dx$

integrating we get $v - \tan\dfrac{v}{2} = x + C$

$\Rightarrow x + y - \tan\left(\dfrac{x+y}{2}\right) = x + c$

i.e. $y = c + \tan\left(\dfrac{x+y}{2}\right)$

18. $y\dfrac{dx}{dy} = y - 1 \Rightarrow \left(1 - \dfrac{1}{y}\right)dy = dx$

Integrating $y - \log y = x + \log c \Rightarrow y - x = \log cy$

i.e. $cy = e^{y-x}$ now put $y(0) = 1$

we get $c = e$

So required solution is $ey = e^{y-x}$ i.e. unique solution.

20. $(y\cos x + 1)\,dx + \sin x\,dy = 0$

Here $M = y\cos x + 1$, $N = \sin x$ so $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i.e. given differential equation is exact so integrating it

$\int (y\cos x + 1)\,dx$ taking y constant

$+ \int \sin x\,dy = c$ taking free from.

$y\sin x + x = c$

21. $x\dfrac{dy}{dx} + y = y^2\log x$ or $\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{1}{xy} = \dfrac{\log x}{x}$

put $\dfrac{1}{y} = v$ i.e. $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

we get $-\dfrac{dv}{dx} + \dfrac{v}{x} = \dfrac{\log x}{x} \Rightarrow \dfrac{dv}{dx} - \dfrac{1}{x}v = \dfrac{\log x}{x}$

So. I.F. is $e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = \dfrac{1}{x}$

24. If $x^h y^k$ be the I.F. then the equation

$(x_y^h k + 2 + 2x^{h+2}y^{k+1})\,dx$

$+ (2x^{h+3}y^k - x^{h+1}\,y^{k+1})dy = 0$ is exact

So $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i.e. $(k+2)\,y^{k+1}x^h + 2x^{h+2}(k+1)\,y^k$

$= 2(h+3)x^{h+2}y^k \ldots\ldots\ (h+1)\,x^h y^{k+1}$

Comparing the coefficient we get

$k + 2 = -h - 1$ and $2k + 2 = 2h + 6$

after solving we get $h = -5/2,\ y = -1/2$

25. The given equation is $(x^2y^2 + xy + 1)y\,dx$ $+ (x^2y^2 - xy + 1)x\,dy = 0$

It is of the form $[f(x,y)]y\,dx + [\phi(x,y)]\,x\,dy = 0$

so its integrating factor is $\dfrac{1}{Mx - Ny}$

i.e. $\dfrac{1}{xy(x^2y^2 + xy + 1) - xy(x^2y^2 - xy + 1)}$

$= \dfrac{1}{2x^2y^2}$

26. The polar sub tangent is $r^2\dfrac{d\theta}{dt} = a$

i.e. $d\theta = \dfrac{a}{r^2}dr$ integrating it

$\theta + c = -\dfrac{a}{r}$ i.e. $r(\theta + c) + a = 0$

27. $y\sin x\,dx + \sin x\,dy = 0$ is exact then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

where $M = y\sin x$, $N = \sin x$

so $\sin x = \cos x \Rightarrow x = \pi/4$

30. Given that $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$, put $y = vx$

i.e. $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ then come get

$$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2 v} = \frac{1+v^2}{2v}$$

i.e. $\frac{2v}{1-v^2}\,dv = \frac{dx}{x}$ integrating it

$$\log(1-V^2) = -\log x + \log c$$

$\Rightarrow \log\{x(1-v^2)\} = \log c$

or $x(1-v^2) = c$ or $x\left[1-\frac{y^2}{x^2}\right] = c$

i.e. $x^2 - y^2 = cx$

33. If $(x^2 + ay + x)dx + (y^2 + x - y)dy = 0$ is exact then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. $a = 1$

40. Given that $(1 + y)xdy - (1 + x)ydx = 0$

i.e. $\left(\frac{1+y}{y}\right)dy - \left(\frac{1+x}{x}\right)dx = 0$

$\left(\frac{1}{y}+1\right)dy - \left(\frac{1}{x}+1\right)dx = 0$ integrating it, we get

$\log y + y - \log x - x = \log c \Rightarrow \log\frac{y}{xc} = x - y$

i.e. $\log\left(\frac{xc}{y}\right) = -x + y \quad \Rightarrow \quad x = \frac{y}{c}e^{y-x}$

$\Rightarrow x = c'y\,e^{y-x}$

41. Put $y - x = v$ i.e. $\frac{dy}{dx} = \frac{dv}{dx} + 1$ in

$\frac{dy}{dx} = 1 + \tan(y - x)$

we get $\frac{dv}{dx} + 1 = 1 + \tan(v) \Rightarrow \cot v\,dv = dx$

i.e. $\log\sin v = x + \log c \Rightarrow \log\left(\frac{\sin(y-x)}{c}\right) = x$

$\Rightarrow \frac{\sin(y-x)}{c} = e^x \Rightarrow \sin(y-x) = ce^x$

42. Put $\frac{y}{x} = v$ i.e. $y = vx$

or $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in $\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$

$v + x\frac{dv}{dx} = \tan v + v \Rightarrow x\frac{dv}{dx} = \tan v$

i.e. $dv\cot v - \frac{dx}{x} = 0$

Integrating it, we get

$\log\sin v - \log x = \log c \Rightarrow \sin v = cx$

i.e. $\sin\frac{y}{x} = cx$

44. Given $\frac{dy}{dx} + \tan x\tan y = \cos x\sec y$

i.e. $\cos y\frac{dy}{dx} + \tan x\,.\,\sin y = \cos x$, put $\sin y = v$

i.e. $\cos y\frac{dy}{dx} = \frac{dv}{dx}$ we get $\frac{dv}{dx} + \tan x\,.\,v = \cos x$

I.F. is $e^{\int\tan x\,dx} = e^{\log\sec x} = \sec x$

So solution is $v\,.\,\sec x = \int\sec x\cdot\cos x\,dx + c$

i.e. $\sin y\cdot\sec x = x + c$

47. Given that $(2x^3 + y)dx - x\,dy = 0$

here $M = 2x^3 + y$, $N = -x$ i.e. $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$

So, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x}(1+1) = -\frac{2}{x} = f(x)$

so I.F. is $e^{-\int\frac{2}{x}dx} = e^{-2\log x} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}$

48. $\frac{dy}{dx} + \tan x.y = \cos x$, $y(0) = 0$

I.F. is $e^{\int\tan x\,dx} = e^{\log\sec x} = \sec x$

so solution is $y\sec x = \int\sec x\cdot\sin\,dx + c = x + c$

i.e. $y\sec x = x + c$

put $y(0) = 0 \Rightarrow 0 = 0 + c \Rightarrow c = 0$

so required solution is $y\sec x = x \Rightarrow y = x\cos x$

52. Given that $x\frac{dx}{dy} = y - 1$ and $x(1) = 0$

i.e. $x\,dx = (y + 1)\,dy$ integrating it $\frac{x^2}{2} = \frac{y^2}{2} - y + c$

put $x(1) = 0$ we get $0 = \frac{1}{2} - 1 + c \Rightarrow c = \frac{1}{2}$

So required solution is $x^2 - y^2 = 2\left(\frac{1}{2} - y\right)$

i.e. a unique solution.

56. Given $y = Ae^{2x} + Be^{-3x} + Ce^x$

differentiate it $y' = 2Ae^{2x} - 3Be^{-3x} + Ce^x$

$$y'' = 4Ae^{2x} + 9Be^{-3x} + Ce^x$$

$$y''' = 8Ae^{2x} - 27Be^{-3x} + Ce^x$$

after solving we get $y''' - 7y' + 6y = 0$

58. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$, put $\frac{dy}{dx} = v$, $\frac{d^2v}{dx^2} = \frac{dv}{dx}$

we get $\frac{dv}{dx} + \frac{2}{x}v = 0$

$\Rightarrow \quad \frac{dv}{v} + \frac{2}{x}dx = 0$ integrating it

$$\log v + 2\log x = \log c \Rightarrow vx^2 = c \Rightarrow \frac{dy}{dx} = \frac{c}{x^2}$$

again integrating, we get $y = -\frac{c}{x} + B = \frac{A}{x} + B$

59. We have $2xydx + (x^2 + 1)dy = 0$

Here, $M = 2xy, N = x^2 + 1$ so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$

i.e. equation is exact, so integrating it

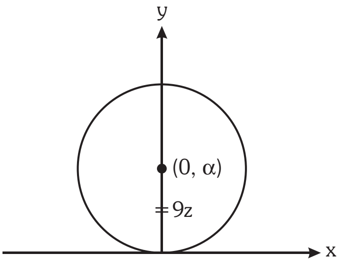
$$\underset{y\text{ is constant}}{\int 2xydx} + \underset{\text{Free from } x}{\int (x^2 + 1)dy} = c$$

$$x^2y + y = c \because y(1) = 2 \text{ we get } c = 4$$

$\therefore \quad y(x^2 + 1) = 4$

61. Put $\frac{y}{x} = v$ we get required solution.

65. Let Co-ordinates of the centre of the circles be $(0, \alpha)$ and their radii r then equation of circle is



$$x^2 + (y - \alpha)^2 = r^2 \qquad \ldots\text{(i)}$$

since it pass through (0, 0) so we have

$$\alpha^2 = r^2$$

So (1) reduces to $x^2 - (y - \alpha)^2 = \alpha^2$

$$\Rightarrow \quad x^2 + y^2 = 2\alpha y \qquad \ldots\text{(ii)}$$

Differentiating it w.r.t. x

$$2x + 2y\frac{dy}{dx} = 2\alpha\frac{dy}{dx}$$

$$2xy + 2y^2\frac{dy}{dx} = (x^2 + y^2)\frac{dy}{dx}$$

i.e., $(y^2 - x^2)\,y' + 2xy = 0$

66. Given $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right) \Rightarrow \frac{dy}{y - ay^2} = \frac{dx}{x + a}$

$$\left(\frac{1}{y} + \frac{a}{1 - ay}\right)dy = \frac{1}{x + a}dx\,,$$

Integrating both sides, we get

$\log y - \log(1 - ay) = \log(x + a) + \log c$

$$\log\left\{\frac{y}{1 - ay}\right\} = \log\{c(x + a)\}$$

$\Rightarrow y = c(1 - ay)(x + a)$

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CHAPTER
2
Linear Differential Equations with Constant Coefficients

DEFINITIONS

1. A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n y = Q$$

is a linear differential equation of order n with constant coefficients. Here $a_1, a_2, \ldots a_n$ are all constants and Q is a function of x.

2. If $y = f(x)$ and $y = g(x)$ are the general solution (Complementary function) and particular solution (P.I.) respectively then $y = f(x) + g(x)$ i.e. $y = \text{C.F.} + \text{P.I.}$

DETERMINATION OF COMPLEMENTARY FUNCTION (C.F.)

Consider a linear differential equation of order n with constant coefficient $f(D)\, y = 0$

i.e. $$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n]\, y = 0 \qquad \ldots(1)$$

then its auxiliary equation (A.E.) is

$m^n + a_1 m^{n-1} + \ldots + a_n = 0$ if it will given n roots say $m_1, m_2 \ldots m_n$ then

Case I. If all the roots $m_1, m_2 \ldots m_n$ are distinct then general solution (C.F.) is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$$

Case II. If two roots $m_1 = m_2$ are equal then C.F. is

$$(C_1 + C_2 x)\, e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$$

Similarly, if three roots $m_1 = m_2 = m_3$ are equal then C.F. is

$$(C_1 + C_2 x + C_3 x)\, e^{m_1 x} + C_4 e^{m_4 x} + \ldots + C_n e^{m_n x}$$

Case III. If two roots $m_1 = \alpha + iB$ and $m_2 = \alpha - iB$ are complex the C.F. is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

or $$C_1 e^{\alpha x} \cos(\beta x + C_2)$$

If the imaginary roots also repeated i.e. $\alpha + i\beta$ and $\alpha - i\beta$ occur twice then C.F. is

$$y = e^{\alpha x}\{(C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x\}$$

Case IV. If a pair of the roots are irrational i.e. $\alpha \pm \sqrt{\beta}$ then C.F. is

$$e^{\alpha x}(C_1 \cos h\, \sqrt{\beta} x + C_2 \sin h \sqrt{\beta} x)$$

or $$C_1 e^{\alpha x} \sin h\, (\sqrt{\beta}\, x + C_2)$$

or $$C_1 e^{\alpha x} \cos h\, (\sqrt{\beta}\, x + C_2)$$

THE PARTICULAR INTEGRAL (P.I.)

Let the linear differential equation of n$^{\text{th}}$ order is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = r(x),\ r(x) \neq 0$$

i.e. $f(D) y = Q(x)$ then its particular integral (P.I.) is $\frac{1}{F(D)} Q(x)$.

Case I. If differential equation is $f(D)\, y = x^m$ then P.I. is $\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$ expending $[f(D)]^{-1}$ in ascending powers of D by using binomial theorem.

Case II. For $f(D) = e^{ax}\phi(x)$

$$\text{P.I. is } \frac{1}{f(D)} e^{ax}\phi(x) = e^{ax} \frac{1}{f(D+a)} \phi(x)$$

Case III. For $f(D) = \sin ax$ or $\cos ax$

then P.I. is $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$, provided $f(-a^2) \neq 0$

and $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$, provided $f(-a^2) \neq 0$

If $f(-a^2) = 0$ then

$\frac{1}{f(D^2)}\sin ax = x \cdot \frac{1}{f'(-a^2)}\sin ax$, similar for $\cos ax$

If $f'(-a^2) = 0$ then

$\frac{1}{f(D^2)}\sin ax = x^2 \frac{1}{f''(-a^2)}\sin ax$, similar for $\cos ax$ and so on.

Case IV. If $f(D) = e^{ax}$ then

$$\begin{aligned}\frac{1}{f(D)}e^{ax} &= \frac{1}{f(a)}e^{ax} \text{ if } f(a) \neq 0\\ &= x\frac{1}{f'(a)}e^{ax} \text{ if } f(a) = 0\\ &= x^2\frac{1}{f''(a)}e^{ax} \text{ if } f'(a) = 0\\ &= e^{ax}\frac{1}{f(D+a)}.1 \text{ if } f(a) = 0\end{aligned}$$

Case V. $\frac{1}{D-a}\phi(x) = e^{ax}\int e^{-ax}\phi(x)\,dx$

Case VI. For $f(D) = xV$, V is any function of x

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}\cdot V$$

Case VII. $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax}\phi(x)\,dx$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The solution of $\frac{d^2y}{dx^2} + 4y = 0$ is

(a) $C_1 \cos x + C_2 \sin x$

(b) $C_1 \cos 2x + C_2 \sin 2x$

(c) $C_1e^{2x} + C_2e^{-2x}$

(d) $C_1e^{x} + C_2e^{-x}$

2. The particular integral of $y'' - 5y' + 6y = \sin 3x$ is

(a) $\frac{1}{12}(\cos 3x - \sin 3x)$

(b) $\frac{1}{12}(5\cos 3x - \sin 3x)$

(c) $\frac{1}{78}(5\cos 3x - \sin 3x)$

(d) $\frac{1}{78}(5\cos 3x + \sin 3x)$

3. The complementary function of $(D^4 - D^2)\,y = 2$ is

(a) $C_1 + C_2x + C_3x^2$ (b) $C_1 + C_2x + C_3e^x$

(c) $(C_1 + C_2x)x^2 + C_3$ (d) $C_1 + C_2x + C_3x^3$

4. The particular integral of $y''' + 3y'' + 3y' + y = e^{-x}$ is

(a) $\frac{1}{6}x\,e^{-x}$ (b) $\frac{1}{6}x^2e^{-x}$

(c) $\frac{1}{6}x^3e^{-x}$ (d) $\frac{1}{6}x^4e^{-x}$

5. The solution of $(D^2 + 1)\,y = 0$ when $y(0) = 2$ and $y(\pi/2) = -2$ is

(a) $\cos x + \sin x$ (b) $2\cos x + 2\sin x$

(c) $2\cos x + \sin x$ (d) $2\cos x - 2\sin x$

6. The particular integral of $(D^2 + 9)y = \cos 4x$ is

(a) $\cos 4x$ (b) $\frac{1}{7}\cos 4x$

(c) $\frac{1}{5}\cos 4x$ (d) $-\frac{1}{7}\cos 4x$

7. The particular integral of $(D^2 - 4D + 4)y = x^2$ is

(a) $x^2 + 2x$ (b) $x^2 + 2x + \frac{3}{2}$

(c) $\frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$ (d) $\frac{1}{3}\left(x^2 + 2x + \frac{3}{2}\right)$

8. The particular integral of $(D^2 - 6D + 9)y = 6e^{3x}$ is

(a) $3e^{3x}$ (b) $6\,e^{3x}$ (c) $3x\,e^{3x}$ (d) $3x^2\,e^{3x}$

9. The particular integral of $(D^3 + 1)\,y = e^{2x}$ is

(a) $\frac{e^x}{3}$ (b) $\frac{e^{2x}}{9}$ (c) $\frac{e^{-x}}{9}$ (d) $\frac{e^{-2x}}{3}$

10. The particular integral of $(D^2 - 5D + 6)y = \cos 3x$ is

(a) $\frac{1}{78}(\cos 3x - \sin 3x)$

(b) $\frac{1}{78}(3\cos 3x - \sin 3x)$

(c) $\frac{1}{78}(-5\sin 3x - \cos 3x)$

(d) $\frac{1}{78}(5\cos 3x + \sin 3x)$

11. $\frac{1}{f(D)} e^{ax}. V$ is

(a) $e^{ax} \frac{1}{f(D)} V$ (b) $e^{ax} \frac{1}{f(D+a)} V$

(c) $e^{ax} \frac{1}{f(D-a)} V$ (d) $e^{ax} V \frac{1}{f(D+a)} V$

12. $\frac{1}{D^2 + 4} \sin 2x$ is equal to

(a) $\frac{x \cos 2x}{2}$ (b) $-\frac{x \cos 2x}{2}$

(c) $\frac{x \cos 2x}{4}$ (d) $-\frac{x \cos 2x}{4}$

13. The solution of $(D^2 + 1)y = 0$ with $y(0) = 0, y'(0) = 0$ is

(a) $y = \cos x$ (b) $y = \sin x$

(c) $y = 0$ (d) Infinite solution

14. The solution of ordinary differential equation of n order contains

(a) n-arbitrary constants

(b) More than n-arbitrary constants

(c) Less then n-arbitrary constants

(d) No arbitrary constants

15. The equation $ydx + xdy = 0$ is

(a) Exact (b) Not-exact

(c) Partial differential equation

(d) Differential equation of order 2

16. The differential equation derived from $y = Ae^x + Be^{-x}$ have the order

(a) 1 (b) 2 (c) 3 (d) Infinite

17. The solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ is

(a) $C_1e^x + C_2e^{-x}$ (b) $(C_1 + C_2x)e^x$

(c) $(C_1 + C_2)e^x$ (d) $C_1 + C_2xe^x$

18. The particular integral of $y'' - 2y' + y = x^2e^x$ is

(a) $\frac{x^4e^x}{4}$ (b) $\frac{x^2e^{2x}}{12}$ (c) $\frac{x^4e^x}{12}$ (d) $\frac{x^2e^{2x}}{4}$

19. The particular integral of $(D^2 + 9)\ y = \cos 2x + \sin 2x$ is

(a) $\frac{1}{5}(\cos 2x + \sin 2x)$ (b) $\frac{1}{3}\cos(2x + \sin 2x)$

(c) $\frac{1}{2}(\cos 2x + \sin 2x)$ (d) None of these

20. The value of $\frac{1}{D^2}\sin 4x$ is

(a) $-\frac{1}{4}\sin 4x$ (b) $-\frac{1}{16}\sin 4x$

(c) $-\frac{1}{16}\cos 4x$ (d) $-\frac{1}{4}\cos 4x$

21. The value of $\frac{1}{D^2 + 3}\sin\sqrt{3}\,x$ is

(a) $-\frac{x}{2\sqrt{3}}\cos\sqrt{3}x$ (b) $\frac{x}{2\sqrt{3}}\cos\sqrt{3}x$

(c) $\frac{x}{2\sqrt{3}}\sin\sqrt{3}x$ (d) $-\frac{x}{2\sqrt{3}}\sin\sqrt{3}x$

22. $\frac{dy}{dx} + Py = Q$ is a linear differential equation of degree first order if

(a) P, Q are functions of x only

(b) P, Q are functions of y only

(c) P and Q are functions of x and y

(d) None of these

23. The differential equation $\frac{dy}{dx} + p(x)\,y = q(x)$ is homogeneous when

(a) $p(x) = 0$ (b) $q(x) \neq 0$

(c) $p(x) \neq 0$ (d) $q(x) = 0$

24. If x^hy^k is an integrating function of $(x^7y^2 + 3y)dx + (3x^8y - x)dy = 0$ then

(a) $h = -7, k = 1$ (b) $h = 7, k = 1$

(c) $h = -7, h = -1$ (d) $h = 7, k = -1$

25. A differential equation of first order and first degree is homogeneous if

(a) $y' = \phi\left(\frac{y}{x}\right)$ (b) $y' = \phi(x)$

(c) $y' =$ constant (d) None of these

26. The differential equation $\frac{dy}{dx} - \frac{1}{x}\cdot y = 2x$ with $y(0) = 0$ has

(a) Unique solution (b) Finite solution

(c) No solution (d) Infinite solutions

27. The solution of $(D^4 + D^2)y = 0$ with $y(0) = y'(0) = y''(0) = 0$, $y'''(0) = 1$ is
(a) $y = \sin x$ (b) $y = x + \sin x$
(c) $y = x - \cos x$ (d) $y = x - \sin x$

28. The particular integral of $(D^2 + 2D + 2)y = \sin hx$ is
(a) $\dfrac{e^x - 5e^{-x}}{10}$ (b) $\dfrac{e^x + e^{-x}}{10}$
(c) $\dfrac{5e^x - e^{-x}}{10}$ (d) None of these

29. $\dfrac{1}{D^2 + a^2} \sin ax$ is equal to
(a) $\dfrac{x}{2a} \sin ax$ (b) $-\dfrac{x}{2a} \sin ax$
(c) $\dfrac{x}{2a} \cos ax$ (d) $-\dfrac{x}{2a} \cos ax$

30. The differential equation of the solution $y = A \cos(\omega x - \theta)$ is
(a) $\dfrac{d^2y}{dx^2} = y$ (b) $\dfrac{d^2y}{dx^2} = 0$
(c) $\dfrac{d^2y}{dx^2} = -\omega^2 y$ (d) None of these

31. The order and degree of $e^{y'''} - y'' + xy = 0$ are
(a) 3, 1 (b) 3, infinite
(c) 3, not defined (d) 3, finite

32. $\dfrac{1}{D^2 - 2D + 2} \sin \sqrt{2}x$ is equal to
(a) $\dfrac{1}{2\sqrt{2}} \sin \sqrt{2}x$ (b) $\dfrac{1}{2\sqrt{2}} \cos \sqrt{2}\, x$
(c) $\dfrac{1}{2} \sin \sqrt{2}x$ (d) $\dfrac{1}{2} \cos \sqrt{2}x$

33. If $\phi(x)$ be any function then $\dfrac{1}{D - a} \phi(x)$ is equal to
(a) $e^{-ax} \int e^{ax} \phi(x)\, dx$ (b) $e^{ax} \int \phi(x)\, dx$
(c) $e^{-ax} \int \phi(x)\, dx$ (d) $e^{ax} \int e^{-ax} \phi(x) dx$

34. The homogeneous differential equation $Mdx + Ndy = 0$ can be reduced to a differential equation in which the variables are separated by the substitution
(a) $y = vx$ (b) $x + y = v$
(c) $xy = v$ (d) $x - y = v$

35. The solution obtained from complete primitive by giving particular values to arbitrary constants is
(a) Complete solution (b) Singular solution
(c) Particular integral (d) Non-singular solution

36. The particular integral of $(D^4 + 4)y = x^4$ is
(a) $\dfrac{x^4 - x^2}{4}$ (b) $\dfrac{x^4 + 4}{4}$
(c) $\dfrac{x^4 - 6}{4}$ (d) $\dfrac{x^4 - x^2 - 6}{4}$

37. Particular integral of $(D^2 + 1)y = x^2$ is
(a) $x^2 + 2$ (b) $x^2 - 2$
(c) $x^2 + 1$ (d) $x^2 - 1$

38. The solution of $(D^3 + 2D^2 + D)y = 0$ is
(a) $y = C_1 + (C_2 + C_3x)e^x$
(b) $y = C_1 + (C_2 + C_3x)e^{-x}$
(c) $y = (C_1 + C_2x + C_3x^2)e^x$
(d) $y = C_1x + C_2x^2 + C_3e^x$

39. $\dfrac{1}{D - 2i} \sec 2x$ is equal to
(a) $\int e^{-2ix} \sec 2x\, dx$ (b) $e^{2ix} \int \sec 2x\, dx$
(c) $e^{-2ix} \int e^{2ix} \sec 2x\, dx$ (d) $e^{2ix} \int e^{-2ix} \sec 2x dx$

40. Which one of the following is not an integrating function for $xy' = y$
(a) $\dfrac{1}{x + y}$ (b) $\dfrac{1}{xy}$ (c) $\dfrac{1}{x^2}$ (d) $\dfrac{1}{y^2}$

41. The solution of $(D^3 + 6D^2 + 12D + 8)y = 0$ with $y(0) = y'(0) = 0$, $y''(0) = 2$ is
(a) $x^2e^{2x} + x + 1$ (b) $x^2e^{-2x} - 2x$
(c) x^2e^{-2x} (d) x^2e^{2x}

42. Integrating factor of $\cos^2 x \dfrac{dy}{dx} + y = \tan x$ is
(a) $e^{\sin x}$ (b) $e^{\sec x}$ (c) $e^{\operatorname{cosec} x}$ (d) $e^{\tan x}$

43. If a equation contains x-arbitrary constants then the order of differential equation derived is
(a) $n - 1$ (b) n (c) $n + 1$ (d) 1

44. $\dfrac{1}{D^2 + 2D + 1} \cdot \dfrac{e^{-x}}{x^2}$ is equal to
(a) $e^x \log x + x^2 + 1$ (b) $-e^{-x} \log x$
(c) $-e^{-x} + \log x + 1$ (d) $e^{-x} - \log x - 1$

45. The solution of the differential equation $(x - y - 1) \dfrac{dy}{dx} = 1$ is
(a) $x - y - z = Ce^y$ (b) $y - z - x = Ce^z$
(c) $z - x - y = Ce^x$ (d) None of these

46. The solution of $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1$ is

(a) xe^{x} (b) $-x\,e^{-x}$ (c) e^{-x} (d) $-e^{-x}$

47. The differential equation whose auxiliary equation has the roots 0, 1, 1 is

(a) $(D^3 - 2D^2 + D)\,y = 0$

(b) $(D^3 + 2D^2 + D)\,y = 0$

(c) $(D^3 - D^2 + 2D)\,y = 0$

(d) $(D^3 + D^2 + 2D)y = 0$

48. The particular integral of $(D^2 - 1)\,y = \cos x$ is

(a) $\frac{1}{2}\cos x$ (b) $-\frac{1}{2}\cos x$

(c) $-\frac{1}{2}\sin x$ (d) None of these

49. The particular integral of $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$ is

(a) $2(\log x)$ (b) $2(\log x)^2$

(c) $2(\log x)^3$ (d) $(\log x)^3$

50. The solution of the differential equation $(D^2 + 1)\,y = 0$ with $y(0) = 1$, $y(\pi/2) = 2$ is

(a) $y = 2\cos x + \sin x$ (b) $y = \cos x + 2\sin x$

(c) $y = \cos x + \sin x$ (d) $y = 2\cos x + 2\sin x$

51. $\frac{1}{f(D)}x^2 \sin ax$ is equal to

(a) $e^{i\,ax}\frac{1}{f(D + i\,a)}x^2$

(b) Real part of $e^{i\,ax}\frac{1}{f(D + ia)}x^2$

(c) $e^{-iax}\frac{1}{f(D - ia)}x^2$

(d) Imaginary part of $e^{iax}\frac{1}{f(D + ia)}x^2$

52. The solution of $y'' + a^2y = 0$ with $y(0) = 0$, $y'(0) = a$ is

(a) $\sin ax$ (b) $\cos ax$ (c) $\sec ax$ (d) $\operatorname{cosec} ax$

53. The solution of $y'' + 2\,y' + 2y = 0$, with $y(0) = 0$, $y'(0) = 1$ is

(a) $e^{x}\sin x$ (b) $e^{-x}\sin x$

(c) $e^{x}\cos x$ (d) $e^{-x}\cos x$

54. $\frac{1}{D^2 - 2D + 2}x$ is equal to

(a) $\frac{x+1}{2}$ (b) $\frac{x}{2}$ (c) $\frac{x-1}{2}$ (d) $-\frac{x}{2}$

55. The particular integral of $(D^2 + 1)y = x^2$ is

(a) $1 + x^2$ (b) $x^2 = 2$ (c) $-x^2 + 2$ (d) $x - 2$

56. The P.I. of $(D - a)^2 y = e^{ax} f''(x)$ is

(a) $e^{-ax}f(x)$ (b) $f'(x)$

(c) $e^{ax}f(x)$ (d) $e^{ax}f'(x)$

57. The particular integral of $(D^2 - 2D + 4)y = e^{x}\cos x$ is

(a) $\frac{1}{2}e^{x}\cos x$ (b) $\frac{1}{2}e^{x}\sin x$

(c) $e^{x}\cos x$ (d) $e^{x}\sin x$

58. $e^{-x}(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x) + C_3e^{2x}$ is the general solution of

(a) $(D^3 - 4)\,y = 0$ (b) $(D^3 - 8)\,y = 0$

(c) $(D^3 + 8)y = 0$ (d) $(D^3 + 4)\,y = 0$

59. $(C_1 + C_2x) + (C_3 + C_4x)e^{-x}$ is the solution of

(a) $(D^4 + D^3 + D^2)y = 0$

(b) $(D^4 + D^2)\,y = 0$

(c) $(D^4 + 2D^3 + D^2)y = 0$

(d) $(D^4 - D^2)\,y = 0$

60. The P.I. of $(D^3 + 1)\,y = (e^{x} + 1)^2$ is

(a) $e^{2x} + e^{x} + 1$ (b) $\frac{e^{2x}}{9} + \frac{e^{x}}{2} + 1$

(c) $e^{2x} - 2e^{x} + 1$ (d) $\frac{e^{2x}}{2} - e^{x} + 1$

61. The P.I. of $(D^2 + D - 6)y = x$ is

(a) $\frac{6x+1}{12}$ (b) $-\frac{6x+1}{18}$

(c) $\frac{6x+1}{18}$ (d) $-\frac{6x+1}{36}$

62. The P.I. of $(D^2 - 4D + 3)y = e^{2x}\sin 3x$ is

(a) $-\frac{e^{2x}\sin 3x}{10}$ (b) $-\frac{e^{2x}\sin 3x}{6}$

(c) $\frac{e^{2x}\sin 3x}{8}$ (d) None of these

63. The differential equation derived from $y = Ae^{x} + Be^{-2x} + Ce^{3x}$ have the order

(a) 1 (b) 2 (c) 3 (d) 4

64. The differential equation derived from $y = Ae^{x} + Be^{-x}$ have the degree

(a) 1 (b) 2 (c) 3 (d) 0

65. The particular integral of $(D-2)^2 y = e^{2x}\, f'''(x)$ is

(a) $e^{2x} f(x)$ (b) $-e^{2x} f'(x)$

(c) $e^{2x}\, f''(x)$ (d) $e^{2x}\, f'(x)$

66. $\dfrac{1}{f(D)} x^3 \cos x$ is equal to

(a) $e^{ix} \dfrac{1}{f\,(D+i)} x^3$

(b) Real part of $e^{ix} \dfrac{1}{f(D+i)} x^3$

(c) Real part of $e^{ix} \dfrac{1}{f(D-i)} x^3$

(d) $e^{ix} \dfrac{1}{f(D+i)} x^3 \cos x$

67. $\dfrac{1}{D^2-2D+1} xe^x \sin x$ is equal to

(a) $e^x\ (x \cos x + \sin x)$

(b) $-e^{-x}\ (x \cos x + \sin x)$

(c) $e^x\ (x \sin x + 2 \cos x)$

(d) $-e^x\ (x \sin x + 2 \cos x)$

68. $\dfrac{1}{D^2-5D+6} x$ is equal to

(a) $\dfrac{6x+5}{36}$ (b) $\dfrac{2x+5}{18}$

(c) $\dfrac{6x-5}{36}$ (d) $\dfrac{2x-5}{18}$

69. The particular integral of $(D^3+8)y = x^4+2x+1$ is

(a) $\dfrac{x^3+x-1}{8}$ (b) $\dfrac{x^4-x+1}{8}$

(c) $\dfrac{x^5+x^4-1}{8}$ (d) $\dfrac{x^4-2x+3}{8}$

70. The solution of $\dfrac{d^2y}{dx^2} = a+bx+cx^2$ with $y' = 0$ when $x = 0$ and $y = d$ when $x = 0$ is

(a) $y = ax + \dfrac{bx^2}{2} + \dfrac{cx^3}{3} + \dfrac{d}{4}$

(b) $y = \dfrac{ax^2}{2} + \dfrac{bx^3}{6} + \dfrac{cx^4}{12} + d$

(c) $y = \dfrac{ax^2}{2} + \dfrac{bx^3}{3} + \dfrac{cx^4}{4} + d$

(d) $y = a + \dfrac{bx}{2} + \dfrac{cx^3}{4} + \dfrac{dx^4}{8}$

71. The solution of $\dfrac{d^2y}{dx^2} - y = 1$, $y(0) = 0$, $y\,(-\alpha) \rightarrow$ finite is

(a) e^x (b) e^x+1 (c) e^x-1 (d) $-e^x-1$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(b)	**4.**	(c)	**5.**	(d)	**6.**	(d)	**7.**	(c)	**8.**	(d)	**9.**	(b)	**10.**	(c)
11.	(b)	**12.**	(d)	**13.**	(c)	**14.**	(a)	**15.**	(a)	**16.**	(b)	**17.**	(b)	**18.**	(c)	**19.**	(a)	**20.**	(b)
21.	(a)	**22.**	(a)	**23.**	(d)	**24.**	(a)	**25.**	(a)	**26.**	(d)	**27.**	(d)	**28.**	(a)	**29.**	(d)	**30.**	(c)
31.	(c)	**32.**	(b)	**33.**	(d)	**34.**	(a)	**35.**	(c)	**36.**	(c)	**37.**	(b)	**38.**	(a)	**39.**	(d)	**40.**	(a)
41.	(c)	**42.**	(d)	**43.**	(b)	**44.**	(b)	**45.**	(a)	**46.**	(c)	**47.**	(a)	**48.**	(b)	**49.**	(c)	**50.**	(b)
51.	(d)	**52.**	(a)	**53.**	(b)	**54.**	(a)	**55.**	(b)	**56.**	(c)	**57.**	(a)	**58.**	(b)	**59.**	(c)	**60.**	(b)
61.	(d)	**62.**	(a)	**63.**	(c)	**64.**	(a)	**65.**	(d)	**66.**	(b)	**67.**	(d)	**68.**	(a)	**69.**	(b)	**70.**	(b)
71.	(c)																		

HINTS AND SOLUTIONS

1. Given $(D^2+4)y = 0$ A.E. is $m^2+4=0$ i.e. $m = \pm\, 2i$

So solution is $y = C_1 \cos 2x + C_2 \sin 2x$

2. Given differential equation is $(D^2-5D+6)y = \sin 3x$.

Its particular integral is $\dfrac{1}{D^2-5D+6} \sin 3x$

$$= \frac{1}{-9-5D+6} \sin 3x = \frac{1}{-5D-3} \sin 3x$$

$$= -\frac{(5D-3)}{(5D+3)\,(5D-3)}\sin 3x$$

$$= -\frac{(5D-3)}{25D^2-9}\sin 3x$$

$$= -\frac{1}{-225-9}(15\cos 3x - 3\sin 3x)$$

$$= \frac{1}{234}(15\cos 3x - 3\sin 3x)$$

$$= \frac{1}{78}(5\cos 3x - \sin 3x)$$

4. $\frac{1}{D^3+3D^2+3D+1}e^{-x}$

$$= e^{-x}\frac{1}{(D-1)^3+3(D-1)^2+3(D-1)+1}.1$$

$$= \frac{e^{-x}}{D^3-1-3D(D-1)+3D^2+3-6D+3D-2}$$

$$= -e^{-x}\cdot\frac{1}{D^3}.1 = \frac{x^3e^{-x}}{3\times 2} = \frac{1}{6}x^3e^{-x}$$

6. $\frac{1}{D^2+9}\cos 4x = \frac{1}{-16+9}\cos 4x = -\frac{1}{7}\cos 4x$

7. $\frac{1}{D^2-4D+4}x^2 = \frac{1}{4}\left(1-D+\frac{D^2}{4}\right)^{-1}\cdot x^2$

$$= \frac{1}{4}\left[1+D-\frac{D^2}{4}+(-1)\frac{(-2)\cdot D^2}{2}\right]x^2$$

Neglecting higher terms

$$= \frac{1}{4}\left[1+D+\frac{3}{4}D^2\right]x^2 = \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$$

9. $\frac{1}{D^3+1}e^{2x} = \frac{1}{8+1}e^{2x} = \frac{1}{9}e^{2x}$

13. Given that $(D^2+1)\,y=0,\ y(0)=0=y'(0)$

A.E. is $m^2+1=0$ i.e., $m=\pm i$ so its solution is

$y = C_1\cos x + C_2\sin x,\ y' = -C_1\sin x + C_2\cos x$

If $y(0)=0 \Rightarrow 0 = C_1$

If $y'(0)=0 \Rightarrow 0 = C_2$ so $y=0$

16. Given $y = Ae^x + Be^{-x},\ y' = Ae^{-x} - Be^{-x}$

Adding we get $y' + y = 2Ae^x$

Again differentiate $y'' + y' = 2Ae^x$

Subtracting we get $y'' - y = 0$ so its order is 2.

17. Given $(D^2-2D+1)y=0$ A.E. is $m^2-2m+1=0$

i.e. $(m-1)^2 = 0 \Rightarrow m = 1, 1$

So solution is

$y = (C_1 + C_2x)e^x$

18. The particular integral is $\frac{1}{D^2-2D+1}e^x\,x^2$

$$= e^x\frac{1}{(D+1)^2-2(D+1)+1}x^2 = e^x\frac{1}{D^2}x^2 = \frac{e^xx^4}{12}$$

24. If x^hy^k be the I.F. of $(x^7y^2+3y)dx$

$$+ (3x^8y - x)dy = 0$$

then $(x^{h+7}y^{k+2} + 3x^hy^{k+1})dx$

$$+ (3x^{h+8}y^{k+1} - x^{h+1}y^k)\,dy = 0$$

is exact i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e., $(k+3)x^{h+7}y^{k+1} + 3(k+1)x^hy^k$

$$= 3\,(h+8)x^{h+7}y^{k+1} - (h+1)\,x^h\,y^k$$

Equating the coefficient, we get

$k+3 = 3h+24$ & $3k+3 = -h-1$

After solving, we get $h = -7, k = 1$

27. Given

$(D^4+D^2)\,y = 0,\ y(0) = y'(0) = y''(0),\ y'''(0) = 1$

A.E. is $m^4+m^2=0$ i.e. $m = 0, 0, +i, -i$

so its solution is $y = C_1 + C_2x + C_3\cos x + C_4\sin x$

so $y' = C_2 - C_3\sin x + C_4\cos x$

$y'' = -C_3\cos x - C_4\sin x,$

$y''' = C_3\sin x - C_4\cos x$

$y(0) = 0 \ \Rightarrow\ C_1 + C_3 = 0$

$y'(0) = 0 \ \Rightarrow\ C_2 + C_4 = 0$

$y''(0) = 0 \ \Rightarrow -C_3 = 0 \Rightarrow C_3 = 0$ so $C_1 = 0$

$y'''(0) = 1 \ \Rightarrow 1 = -C_4 \Rightarrow C_4 = -1 \therefore C_2 = 1$

So the required solution is

$y = x - \sin x$

28. $\frac{1}{D^2+2D+2}\sin hx = \frac{1}{D^2+2D+2}\left(\frac{e^x-e^{-x}}{2}\right)$

$$= \frac{1}{2}\frac{1}{(D^2+2D+2)}e^x - \frac{1}{2}\frac{1}{(D^2+2D+2)}e^{-x}$$

$$= \frac{1}{2}\cdot\frac{1}{5}e^x - \frac{1}{2}\cdot\frac{1}{1}e^{-x} = \frac{e^x-5e^{-x}}{10}$$

30. $y = A\cos(\omega x - \theta)$, $\frac{dy}{dx} = -Aw\sin(\omega x - \theta)$

$$\frac{d^2y}{dx^2} = -A\omega^2\cos(\omega x - \theta)$$

so differential equation is $\frac{d^2y}{dx^2} = -\omega^2 y$

32. $$\frac{1}{D^2 - 2D + 2}\sin\sqrt{2}\,x = \frac{1}{-2 - 2D + 2}\sin\sqrt{2}x$$

$$= -\frac{1}{2D}\sin\sqrt{2}x$$

$$= -\frac{D}{2D^2}\sin\sqrt{2}x = -\frac{1}{2(-2)}\sqrt{2}\cos\sqrt{2}\,x$$

$$= \frac{1}{2\sqrt{2}}\cos\sqrt{2}x$$

36. $$\frac{1}{D^4 + 4}x^4 = \frac{1}{4}\left(1 + \frac{D^4}{4}\right)^{-1}x^4$$

$$= \frac{1}{4}\left(1 - \frac{D^4}{4} + \dots\right)x^4$$

$$= \frac{1}{4}\left(x^4 - \frac{24}{4}\right) = \frac{1}{4}(x^4 - 6)$$

41. Given that $(D^3 + 6D^2 + 12D + 8)\ y = 0$, $y(0) = y'(0) = 0$, $y''(0) = 2$

A.E. is $m^3 + 6m^2 + 12m + 8 = 0$

$$m^2(m + 2) + 4m(m + 2) + 4(m + 2) = 0$$

$$\Rightarrow \quad (m^2 + 4m + 4)(m + 2) = 0$$

$$m = -2, \frac{-4 \pm \sqrt{16 - 16}}{2} = -2, -2, -2$$

So solution is $y = (C_1 + C_2x + C_3x^2)\,e^{-2x}$

$$y' = -2C_1\,e^{-2x} + C_2\,(e^{-2x} - 2x\,e^{-2x}$$

$$+ C_3\,(2x\,e^{-2x} - 2x^2\,e^{-2x})$$

$$y'' = 4C_1\,e^{-2x} + C_2\,(-4e^{-2x} + 4xe^{-2x})$$

$$+ C_3(-8xe^{-2x} + 4x^2e^{-2x} + 2e^{-2x})$$

$$y(0) = 0 \Rightarrow 0 = C_1 \Rightarrow C_1 = 0$$

$$y'(0) = 0 \Rightarrow 0 = -2C_1 + C_2 \Rightarrow C_2 = 0$$

$$y''(0) = 2 \Rightarrow 2 = 4C_1 - 4C_2 + 2C_3 \Rightarrow C_3 = 1$$

So required solution is x^2e^{-2x}.

45. Given that $(x - y - 1)\frac{dy}{dx} = 1 \Rightarrow \frac{dx}{dy} = x - y - 1$

i.e. $\frac{dx}{dy} - x = -(y + 1)$ I.F. is $e^{\int -1\,dy} = e^{-y}$

So its solution is $xe^{-y} = -\int e^{-y}(y + 1)dy + C$

$$xe^{-y} = +(y + 1)\,e^{-y} - \int e^{-y}dy + C$$

$$= (y + 1)\,e^{-y} + e^{-y} + C$$

$$xe^{-y} = (y + 2)\,e^{-y} + C \Rightarrow (x - y - z)\,e^{-y} = C$$

i.e., $\quad x - y - z = Ce^{y}$

49. Given that $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

i.e. $\quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$

put $x = e^2$ we get $[D(D - 1) + D]\,y = 12z$, where $Z = \log x$ & $D \equiv \frac{d}{dz}$

Thus, we have $D^2y = 12z$ so P.I. is $\frac{1}{D^2}12z$

$$= \frac{12\,.\,z^3}{6} = 2z^3 = 2(\log x)^3$$

55. Given P.I. is $\frac{1}{D^2 + 1}x^2 = (1 + D^2)^{-1}\cdot x^2$

$$= (1 - D^2 + \dots)\,x^2 = x^2 - 2$$

56. $$\frac{1}{(D - a)^2}e^{ax}f''(x) = e^{ax}\frac{1}{(D + a - a)^2}f''(x)$$

$$= e^{ax}\frac{1}{D^2}f''(x) = e^{ax}f(x)$$

57. $$\frac{1}{(D^2 - 2D + 4)}e^{x}\cos x$$

$$= e^{x}\frac{1}{(D + 1)^2 - 2(D + 1) + 4}\cos x$$

$$= e^{x}\frac{1}{D^2 + 3}\cos x = e^{x}\frac{1}{-1 + 3}\cos x = \frac{e^{x}\cos x}{2}$$

58. Consider $(D^3 - 8)\,y = 0$, A.E. is $m^3 - 8 = 0$

$(m - 2)(m^2 + 4 - 2m)$ i.e. $m = 2, \frac{2 \pm \sqrt{4 - 16}}{2}$

i.e. $\quad m = 2, \frac{2 \pm 2\sqrt{3}i}{2} = 2, 1 \pm \sqrt{3}i$

So solution is $e^{-x}(C_1\cos\sqrt{3}x$

$$+ C_2\sin\sqrt{3}x) + C_3e^{2x}$$

60. $$\frac{1}{(D^3 + 1)}(e^{x} + 1)^2 = \frac{1}{(D^3 + 1)}(e^{2x} + e^{x} + 1)$$

$$= \frac{1}{D^3+1}\,e^{2x} + \frac{1}{D^3+1}\,e^{x} + \frac{1}{D^3+1} - 1$$

$$= \frac{e^{2x}}{9} + \frac{e^{x}}{2} + (1+D^3)^{-1} - 1 = \frac{e^{2x}}{9} + \frac{e^{x}}{2} + 1$$

62. $\frac{1}{D^2-4D+3}\,e^{2x}\sin 3x$

$$= e^{2x}\,\frac{1}{(D+2)^2 - 4(D+2) + 3}\sin 3x$$

$$= e^{2x}\,\frac{1}{D^2-1}\sin 3x = e^{2x}\,\frac{1}{-9-1}\sin 3x$$

$$= -\frac{e^{2x}\sin 3x}{10}$$

63. $y = Ae^{x} + B\,e^{-2x}$ differentiating it

$$\frac{dy}{dx} = Ae^{x} - 2Be^{-2x}$$

Subtracting $\frac{dy}{dx} - y = -3\,Be^{-2x}$ …(1)

Again differentiating $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6Be^{-2x}$ …(2)

Multiply equation (1) by 2 and add with equation (2), we get

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \text{ so its degree is 1}$$

65. $\frac{1}{(D-2)^2}\,e^{2x}\,f'''(x) = e^{2x}\,\frac{1}{(D+2-2)^2}\,f'''(x)$

$$= \frac{e^{2x}}{D^2}\,f'''(x) = e^{2x}\cdot f'(x)$$

66. $\because \cos x =$ Real part of e^{ix} so

$$\frac{1}{f(D)}\,x^3\cos x = \text{Real part of } e^{ix}\,\frac{1}{f(D+i)}\,x^3$$

67. $\frac{1}{D^2-2D+1}\,x\,e^{x}\sin x = \frac{1}{(D-1)^2}\,(e^{x}\cdot x\sin x)$

$$= e^{x}\,\frac{1}{(D+1-1)^2}\,x\sin x = e^{x}\,\frac{1}{D^2}\,x\sin x$$

$$= e^{x}\left[x\,\frac{1}{D^2}\sin x - \frac{1}{D^2}\left\{2D\left(\frac{1}{D^2}\sin x\right)\right\}\right]$$

$$= e^{x}\left\{-x\sin x - \frac{2}{D^3}\sin x\right\}$$

$$= -e^{x}\,(x\sin x + 2\cos x)$$

68. $\frac{1}{D^2-5D+6}\,x = \frac{1}{6}\left(1 - \frac{5D-D^2}{6}\right)^{-1} x$

$$= \frac{1}{6}\left(1 - \frac{5D-D^2}{6} + \ldots\right) = \frac{1}{6}\left(x + \frac{5}{6}\right) = \frac{6x+5}{36}$$

69. Particular integral is $\frac{1}{D^3+8}\,x^4 + 2x + 1$

$$= \frac{1}{8}\left\{1 + \frac{D^3}{8}\right\}^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \ldots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}(x^4 + 2x + 1) - \frac{1}{8}D^3(x^4 + 2x + 1)$$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x) = \frac{1}{8}(x^4 - x + 1)$$

70. Given $\frac{d^2y}{dx^2} = a + bx + cx^2$, $y'(0) = 0$, $y(0) = d$

Integrating $\frac{dy}{dx} = ax + \frac{bx^2}{2} + \frac{cx^3}{3} + e$

Again integrating

$y = \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12} + ex + f$ …(1)

$$y(0) = d \Rightarrow f = d$$

$$y'(0) = 0 \Rightarrow e = 0$$

put these values in equation (1), we get

$$y = \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12} + d$$

71. Given the $(D^2-1)\,y = 1$, $y(0) = 0$, $y(-\alpha) \to$ finite

A.E. is $m^4 - 1 = 0 \Rightarrow m = \pm 1$

C.F. is $C_1e^{x} + C_2e^{-x}$

P.I. $\frac{1}{D^2-1}\,1 = -(1-D^2)^{-1}\,.1$

$$= -(1 + D^2 + \ldots).\,1 = -1$$

So, solution is $y = C_1e^{x} + C_2\,e^{-x} - 1$ …(1)

$y(0) = 0 \Rightarrow 0 = C_1 + C_2 - 1 \Rightarrow C_1 + C_2 = 1$

$y(-\alpha) = \text{finite} \Rightarrow C_2 = 0$ so $C_1 = 1$

put these values in equation (1) we get

$$y = e^{x} - 1$$

* * *

CHAPTER
3
Homogeneous Linear Differential Equations

DEFINITIONS

1. A differential equation of the form
$$x^n \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = P$$
where $a_1, a_2 \ldots a_n$ are constants and P is either a constant or a function of x is called a homogeneous linear differential equation.

2. To solve homogeneous linear differential equation, we put $x = e^z$ or $\log x = z$ such that $\frac{1}{x} = \frac{dz}{dx}$

   Then $\quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

   So $\quad x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ where $D \equiv \frac{d}{dz}$

   Again
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$
$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

   or $\quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D(D-1)y$

   Similarly, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on. Thus, the new transformed equation is $f(D)y = Q$, where $D \equiv \frac{d}{d^2}$ and Q is a function of z. The general solution of this equation is the sum of particular solution of this equation and general solution of $f(D)\,y = 0$.

3. Now we discuss all the cases to find the complementary function (C.F.) of, $f(D)y = 0$.

   Case I. If $m_1, m_2, \ldots m_n$ are distinct roots of the auxiliary equation of $f(D)y = 0$ then C.F. of $f(D)\,y = Q$ is
$$y = C_1e^{m_1z} + C_2e^{m_2z} + \ldots\ldots C_ne^{m_n \cdot z}$$
   or $y = C_1\,x^{m_1} + C_2x^{m_2} + \ldots\ldots C_nx^{m_n}$

   Case II. If r roots are equal say m and others $(n-r)$ are different then C.F. is
$$(C_1 + C_2z + \ldots\ldots C_r\,z^{r-1})\,e^{mz} + C_{r+1}e^{m_{r+1}z} + \ldots + C_ne^{m_nz}$$
   i.e. $[C_1 + C_2\log x + \ldots + C_r(\log x)^{r-1}]\,x^m + C_{r+1}\,x^{m_{r+1}} + \ldots C_nx^{m_n}$

   Case III. If the roots are imaginary i.e. $\alpha \pm i\beta$ then

   C.F. is $e^{\alpha x}(C_1\cos\beta z + C_2\sin\beta z)$

   $= x^{\alpha}\,[C_1\cos(\beta\log x) + C_2\sin(\beta\log x)]$ or we can write $C_1e^{\alpha z}\cos(\beta z + C_2)$

   i.e. $\quad C_1x^{\alpha}\cos(\beta\log x + C_2)$

   Case IV. If $\alpha \pm i\beta$ occur r times then C.F. is
$$e^{\alpha z}\,[(C_1 + C_2z + \ldots C_r\,z^{r-1})\cos\beta z + (d_1 + d_2z + \ldots d_rz^{r-1})\sin\beta z]$$
   i.e.
$$x^{\alpha}\,[\{C_1 + C_2\log x + \ldots + C_r(\log x)^{r-1}\}\cos(\beta\log x) + \{d_1 + d_2\log x + \ldots + dr(\log x)^{r-1}\}\sin(\beta\log x)]$$

4. Now we discuss the cases to find the particular integral (P.I.) of $f(D)y = Q$

   Case I. $\frac{1}{f(D)}Q = \frac{1}{f(D)}e^{az} = \frac{1}{f(a)}e^{az}$, provided $f(a) \neq 0$

Case II. $\frac{1}{f(D^2)} \cos az = \frac{1}{f(-a^2)} \cos az$

and $\frac{1}{f(D^2)} \sin az = \frac{1}{f(-a^2)} \sin az$, provided $f(-a^2) \neq 0$

Case III. $\frac{1}{f(D)} z^m = [f(D)]^{-1} z^m$ and then expend $[f(D)]^{-1}$ in ascending powers of D.

Case IV. $\frac{1}{f(D)} e^{az} V = e^{az} \frac{1}{f(D+a)} V$

where V is a function of z.

Case V. $\frac{1}{f(D)} (zV) = z \frac{1}{f(D)} V + \left\{ \frac{d}{dD} \cdot \frac{1}{f(D)} \right\} V$

5. If the differential equation is $(a+bx)^n \frac{d^ny}{dx^n} + a_1 (a+bx)^{n-1} . \frac{d^{n-1}y}{dx^{n-1}} + \ldots . a_n y = X$ then it can be reduced to linear differential equation with constant coefficient by substitute $a + bx = e^z$.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The complementary function of $(x^2D^2 + 3xD + 1)\, y = \frac{1}{(1-x)^2}$ is
   - (a) $C_1 + C_2 \log x$
   - (b) $C_1 x + C_2 x \cdot \log x$
   - (c) $\frac{1}{x} (C_1 + C_2 \log x)$
   - (d) $\frac{C_1}{x} + C_2 x \log x$

2. The complementary function of $(x^2D^2 - 4xD + 6)\, y = x^2$ is
   - (a) $C_1 x + C_2 x^2$
   - (b) $C_1 + C_2 x$
   - (c) $C_1 x + C_2 x^3$
   - (d) $C_1 x + C_2$

3. For solving the homogeneous linear equations we substitute $x = e^z$ and $D \equiv \frac{d}{dz}$ than $x^2 \frac{d^2y}{dx^2}$ is
   - (a) $D^2 y$
   - (b) $(D-1)y$
   - (c) $D(D+1)y$
   - (d) $D(D-1)y$

4. The solution of $(x^2D^2 + xD + 1)y = 0$ is
   - (a) $C_1 \cos(\log x) + C_2 \sin(\log x)$
   - (b) $C_1 \cos x + C_2 \sin x$
   - (c) $(C_1 + C_2 x) \cos(\log x)$
   - (d) $(C_2 + C_2 x) \cdot \sin(\log x)$

5. The solution of $(x^2D^2 - xD)\, y = 0$ is
   - (a) $C_1 x + C_2 x^2$
   - (b) $C_1 + C_2 x$
   - (c) $C_1 + C_2 x^2$
   - (d) $C_1 x + C_2 x^3$

6. On putting $x = e^z, x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x$ is transformed to
   - (a) $\frac{d^2y}{dz^2} - y = e^z$
   - (b) $\frac{d^2y}{dz^2} + y = e^z$
   - (c) $\frac{d^2y}{dz^2} + \frac{dy}{dz} + y = e^z$
   - (d) $\frac{d^2y}{dz^2} + \frac{dy}{dz} = e^z$

7. The particular integral of $(x^2D^2 + xD + 1)y = \sin(\log x^2)$ is
   - (a) $\frac{1}{3} \cos(\log x^2)$
   - (b) $-\frac{1}{3} \cos(\log x^2)$
   - (c) $\frac{1}{3} \sin(\log x^2)$
   - (d) $-\frac{1}{3} \sin(\log x^2)$

8. The P.I. of $(x^2D^2 + xD + 1)y = x$ is
   - (a) x
   - (b) $\frac{x}{3}$
   - (c) $\frac{x}{2}$
   - (d) $x^2 + \frac{x}{3}$

9. The complementary function of $(x^2D^2 - 3xD + 4)y = 2x^2$ is
   - (a) $(C_1 + C_2)\, x^2$
   - (b) $(C_1 + C_2 \log x)\, x^2$
   - (c) $C_1 x + C_2 x^2 \log x$
   - (d) $C_1 + C_2 x \log x$

10. The particular integral of $(x^2D^2 + xD - 1)y = x^2 e^x$ is
    - (a) xe^x
    - (b) $(1-x)\, e^x$
    - (c) $\left(1 - \frac{1}{x}\right) e^x$
    - (d) $\left(x - \frac{1}{x}\right) e^x$

11. If we substitute $x = e^z$ in $(x^2D^2 + 3xD + 1)\, y = 0$ then the transformed differential equation is
    - (a) $(D'^2 + 1)y = 0$
    - (b) $(D'^2 + 2)y = 0$
    - (c) $(D' + 1)^2 y = 0$
    - (d) $(D' + 2)^2 y = 0$

12. The P.I. of $(x+a)^2 D^2 y - 4(x+a)Dy + 6y = x$ is
    - (a) $x + a$
    - (b) $\frac{x}{2} + a$
    - (c) $x + \frac{a}{3}$
    - (d) $\frac{x}{2} + \frac{a}{3}$

13. If we substitute $5+2x=e^z$ in the differential equation $(5+2x)^2\dfrac{d^2y}{dx^2}-6(5+2x)\dfrac{dy}{dx}+8y=0$ and $D\equiv\dfrac{d}{dz}$ then it transform to

(a) $(2D^2-3D+1)y=0$ (b) $(D^2-4D+2)y=0$

(c) $(2D^2+3D+1)y=0$ (d) $(2D^2+4D+2)y=0$

14. To transform $(x^2D^2+xD+1)y=0$ into the linear differential equation with constant coefficients, we put

(a) $x=\log z$ (b) $x=e^z$

(c) $x=z^2$ (d) $x=e^{2z}$

15. The P.I. of $(x^3D^3+3x^2D^2+xD+1)y=x$ is

(a) x^2+2x (b) x^2-x+1

(c) $\dfrac{x}{2}$ (d) x^2+x

16. The P.I. of $(x^2D^2-xD-3)y=x^2\log x$ is

(a) $x^2(3\log x+2)$ (b) $-\dfrac{x^2}{3}(3\log x+2)$

(c) $-\dfrac{x^2}{9}(3\log x+2)$ (d) $-x(3\log x+2)$

17. If we substitute $x=e^z$ and $D\equiv\dfrac{d}{dz}$ in $x^3\dfrac{d^3y}{dx^3}+3x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$ then transformed equation is

(a) $(D^3+3D^2+D+1)y=0$

(b) $(D^3+D+1)y=0$

(c) $(D^3+4)y=0$

(d) $(D^3+1)y=0$

18. The particular integral of $(x^2D^2+1)y=3x^2$ is

(a) x (b) x^2 (c) $3x^2$ (d) $2x^2$

19. The solution of $(x^2D^2-4xD+6)y=0$ is

(a) $C_1x+C_2x^2$ (b) $C_1x^2+C_2x^3$

(c) C_1+C_2x (d) $C_1x+C_2x^4$

20. The particular integral of $(x^2D^2-3xD+5)y$ $=x^2\sin(\log x)$ is

(a) $-\dfrac{x^2}{2}\log x\cos(\log x)$

(b) $-\sin x(\log x)$

(c) $-\dfrac{1}{x}\log x$

(d) $-\dfrac{1}{x}\sin x(\log x)$

21. $\dfrac{1}{f(D)}e^x\phi(x)$ is equal to

(a) $e^x\dfrac{1}{f(D+1)}\phi(x)$ (b) $e^x\dfrac{1}{f(D)}\phi(x)$

(c) $e^{-x}\dfrac{1}{f(D)}\phi(x)$ (d) $e^x\dfrac{1}{f(D-1)}\phi(x)$

22. The n^{th} order linear differential homogeneous equation have

(a) Finite singular solution

(b) n-singular solution

(c) No singular solution

(d) Unique singular solution

23. The particular integral of $(x^2D^2-4xD+6)y=\dfrac{42}{x^4}$ is

(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x^3}$ (c) $\dfrac{1}{x^4}$ (d) $\dfrac{1}{x^5}$

24. If we substitute $x=e^z$ in $(x^2D^2+4xD+2)y=0$ then the transformed differential equation is

(a) $(D'^2+3D'+2)y=0$ (b) $(D'^2-3D'+2)y=0$

(c) $(D'^2-3D'+4)y=0$ (d) $(D'^2+3D'+4)y=0$

25. The particular integral of $(x^2D^2-4xD+6)y=x^4$ is

(a) x^4+x^2-1 (b) x^2-x

(c) $\dfrac{x+1}{12}$ (d) $\dfrac{x^4}{2}$

26. The P.I. of $(x^3D^3+2x^2D^2+2)y=10x$ is

(a) $10x$ (b) $20x$ (c) $5x$ (d) $15x$

27. The P.I. of $(x^2D^2-xD+2)y=x\log x$ is

(a) $\left(\dfrac{2\log x-3}{4}\right)x$ (b) $\left(\dfrac{2\log x+3}{4}\right)x$

(c) $\dfrac{2\log x-3}{4}$ (d) $x\log x$

28. If D stands for $\dfrac{d}{dz}$ and $x=e^z$ then $x^3\dfrac{d^3y}{dx^3}$ is

(a) $D(D-1)(D-2)y$

(b) $D(D-1)(D-1)(D-3)y$

(c) $D^3(D-1)y$

(d) $D(D-1)y$

29. The particular integral of $(x^3D^3+2x^2D^2+3xD-3)y=x^2$ is

(a) $\frac{x^2+1}{5}$ (b) $\frac{x^2}{7}$

(c) $\frac{x^3+7x+3}{7}$ (d) $\frac{x^2-3}{7}$

30. $\frac{1}{D-a}Q$ is equal to, where D stands for $\frac{d}{dz}$

(a) $e^{-az}\int e^{az}Qdz$ (b) $\int e^{-az}Qdz$

(c) $\int e^{az}Q$ (d) $e^{az}\int e^{-az}Qdz$

31. The particular integral of

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\cos\log(1+x) \text{ is}$$

(a) $\sin\log(1+x)$

(b) $2\log(1+x)\sin\log(1+x)$

(c) $\log(1+x)\sin\log(1+x)$

(d) $\log(1+x)\cos\log(1+x)$

32. By substitution $y=z^2$, the homogeneous equation of

$$2x^2y\frac{d^2y}{dx^2}+4y^2-x^2\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)=0 \text{ is}$$

(a) $(x^2D^2-xD+1)\,z=0$

(b) $(x^2D^2+1)z=0$

(c) $(x^2D^2-2xD-1)z=0$

(d) $(x^2D^2-2)z=0$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(c)	**3.**	(d)	**4.**	(a)	**5.**	(c)	**6.**	(b)	**7.**	(d)	**8.**	(c)	**9.**	(b)	**10.**	(c)
11.	(c)	**12.**	(d)	**13.**	(b)	**14.**	(b)	**15.**	(c)	**16.**	(c)	**17.**	(d)	**18.**	(c)	**19.**	(b)	**20.**	(a)
21.	(a)	**22.**	(c)	**23.**	(c)	**24.**	(a)	**25.**	(d)	**26.**	(c)	**27.**	(d)	**28.**	(a)	**29.**	(b)	**30.**	(d)
31.	(c)	**32.**	(a)																

HINTS AND SOLUTIONS

1. Given differential equation is $(x^2D^2+3xD+1)y=\frac{1}{(1-x)^2}$ put $x=e^z$ and $D\equiv\frac{d}{dz}$ we get

$$[D(D-1)+3D+1]y=\frac{1}{(1-e^z)^2}$$

i.e. $(D^2+2D+1)y=\frac{1}{(1-e^z)^2}$

A.E. is $m^2+2m+1=0$ i.e. $m=-1,-1$

So, C.F. is $(C_1+C_2z)e^{-z}=(C_1+C_2\log x)\cdot\frac{1}{x}$

4. Given that $(x^2D^2+xD+1)y=0$, put $x=e^z$, $D'\equiv\frac{d}{dz}$

We get $[D'(D'-1)+D'+1]y=0$ i.e. $(D'^2+1)y=0$

A.E. is $m^2+1=0 \Rightarrow m=i,-i$

So solution is $C_1\cos z+C_2\sin z=C_1\cos(\log x)+C_2\sin(\log x)$

6. Given $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x$, put $x=e^z$

$$[D(D-1)+D+1]\,y=e^z$$

$$\Rightarrow \quad (D^2+1)y=e^z$$

7. Given $(x^2D^2+xD+1)y=\sin(\log x^2)$ put $x=e^z$, $D'=\frac{d}{dz}$ we get

$$[D'(D'-1)+D'+1]y=\sin(2z)$$

i.e. $(D'^2+1)y=\sin 2z$

so its P.I. is

$$\frac{1}{(D'^2+1)}\sin 2z=+\frac{1}{-4+1}\sin 2z=-\frac{1}{3}\sin 2z$$

$$=-\frac{1}{3}\sin(2\log x)=-\frac{1}{3}\sin(\log x^2)$$

10. Given differential equation is $(x^2D^2+xD-1)y=x^2e^x$ put $x=e^z$, $D'\equiv\frac{d}{dz}$, we get

$$(D'(D'-1)+D'-1)y=e^{2z}ee^z$$

$$\Rightarrow \qquad (D'^2 - 1)y = e^{2z}\ ee^z = x^2e^x$$

P.I. is $\dfrac{1}{D'^2 - 1}\, x^2e^x = \dfrac{1}{D' + 1}\left(\dfrac{1}{D' - 1}\, x^2e^x\right)$

$$= \frac{1}{D' + 1}\,(x \int x^2e^x\ x^{-1-1}dx) = \frac{1}{D' + 1}\, xe^x$$

$$= x^{-1} \int xe^x\ x^{1-1}dx = x^{-1}\,(x - 1)e^x = e^x - \frac{e^x}{x}$$

$$= \left(1 - \frac{1}{x}\right)e^x$$

12. Given that $(x + a)^2D^2y - 4\,(x + a)Dy + 6y = x$ put $x + a = e^z$ we get

$$[D'(D' - 1) - 4\,(D') + 6]\ y = e^z - a$$

i.e. $\qquad (D'^2 - 5\,D' + 6)y = e^z - a$

So, P.I. is $\dfrac{1}{(D'^2 - 5D' + 6)}\, e^z - a\, \dfrac{1}{D'^2 - 5\,D' + 6}\, .1$

$$= \frac{1}{2}\, e^z - \frac{a}{6}\left(1 - \frac{5D'}{6} + \frac{D'^2}{6}\right)^{-1} .1$$

$$= \frac{e^z}{2} - \frac{a}{6} = \frac{x + a}{2} - \frac{a}{6} = \frac{x}{2} + \frac{a}{3}$$

13. Given that $(2x + 5)^2\, \dfrac{d^2y}{dx^2} - 6(2x + 5)\, \dfrac{dy}{dx} + 8y = 0$

put $2x + 5 = e^z$ we get $[4D'(D' - 1) - 12D' + 8]y = 0$

$\Rightarrow [(D'^2 - D') - 3\,D' + 2]y = 0$

i.e. $(D'^2 - 4D' + 2)y = 0$

16. Differential equation is $(x^2D^2 - xD - 3)y = x^2 \log x$

put $x = e^z$, $D' \equiv \dfrac{d}{dz}$, we get

$[D'(D' - 1) - D' - 3]y = ze^{2z}$

$\Rightarrow\ (D'^2 - 2D' - 3)y = ze^{2z}$

P.I. is $\dfrac{1}{(D'^2 - 2D' - 3)}\, ze^{2z}$

$$= e^{2z}\, \frac{1}{(D' + 2)^2 - 2\,(D' + 2) - 3}\, z$$

$$= \frac{e^{2z}}{D'^2 + 2D' - 3}\, z^2 = -\frac{e^{2z}}{3}\left(1 - \frac{2D'}{3} - \frac{D'^2}{3}\right)^{-1} z$$

$$= -\frac{e^{2z}}{3}\left(1 + \frac{2D'}{3} + \ldots\right)z = -\frac{e^{2z}}{3}\left(z + \frac{2}{3}\right)$$

$$= -\frac{x^2}{3}\left[\log x + \frac{2}{3}\right] = -\frac{x^2}{9}\,(3 \log x + 2)$$

23. Given homogeneous equation is $(x^2D^2 - 4xD + 6)y = \dfrac{4z}{x^4}$ put $x = e^z$ and $D' \equiv \dfrac{d}{dz}$ we get

$[D'(D' - 1) - 4\,D' + 6]\,y = 4ze^{-4z}$

$\Rightarrow \qquad (D'^2 - 5D' + 6)y = 4ze^{-4z}$

P.I. is $\dfrac{1}{D'^2 - 5D' + 6}\, 4ze^{-4z}$

$$\Rightarrow \qquad = 4ze^{-4z}\, \frac{1}{(D' - 4)^2 - 5(D' - 4) + 6} .1$$

$$= 4z^{e-4z}\, \frac{1}{D'^2 - 13D' + 4z} .1$$

$$= \frac{4z}{4z}\, e^{-4z}\left(1 - \frac{13D'}{4z} + \frac{D'^2}{4z}\right)^{-1} .1$$

$$= \frac{4z}{4z}\, e^{-4z}\left(1 + \frac{13D'}{4z} + \ldots\right).1 = e^{-4z} = \frac{1}{x^4}$$

25. Given $\qquad (x^2D^2 - 4xD + 6)y = x^4$, $\qquad$ put $x = e^z$, $D' \equiv \dfrac{d}{dz}$

We have $[D'(D' - 1) - 4D' + 6]\ y = e^{4z}$

i.e. $\ (D'^2 - 5D' + 6)y = e^{4z}$

P.I. $\dfrac{1}{(D'^2 - 5D' + 6)}\, e^{4z} = \dfrac{1}{2}\, e^{4z} = \dfrac{x^4}{2}$

27. Given differential equation is $(x^2D^2 - xD + 2)y = x \log x$

put $x = e^z$, $D' = \dfrac{d}{dz}$

we get $\ (D'(D' - 1) - D' + 2)y = ze^z$

i.e., $\qquad (D'^2 - 2D' + 2)y = ze^z$

P.I. is $\dfrac{1}{D'^2 - 2D' + 2}\, ze^z$

$$= e^z\, \frac{1}{(D' + 1)^2 - 2\,(D' + 1) + 2}\, z$$

$$= e^z\, \frac{1}{D'^2 + 1}\, z = e^z(1 + D'^2)^{-1} \cdot 2 = ze^z$$

$= x \log x$

29. Given $(x^3D^3 + 2\,x^2D^2 + 3xD - 3)y = x^2$

put $x = e^z$ & $D' \equiv \dfrac{d}{dz}$, we get

$[D'(D' - 1)\,(D' - 2) + 2D'(D' - 1) + 3D' - 3]\ y = e^{2z}$

or $(D' - 1)\,(D'^2 + 3)y = e^{2z}$

$$\text{So P.I.}\ \frac{1}{(D'-1)(D'^2+3)}e^{2z} = \frac{e^{2z}}{(2-1)(4+3)} = \frac{e^{2z}}{7}$$

$$= \frac{x^2}{7}$$

31. Given differential equation is

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\cos\log(1+x)$$

Put $1 + x = e^z$, we get

$$[D(D-1) + D + 1]y = 2\cos z$$

$$\Rightarrow \qquad (D^2+1)y = 2\cos z$$

$$D \equiv \frac{d}{dz}$$

The particular integral is $\frac{1}{D^2+1}2\cos z$

$$= \frac{2z}{2}\sin z = 2\sin z = \log(1+x)\sin\log(1+x)$$

32. Given differential equation is

$$2x^2y\frac{d^2y}{dx^2} + 4y^2 - x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 0$$

Put $y = z^2, \frac{dy}{dx} = 2z\frac{dz}{dx}$,

$$\frac{d^2y}{dx^2} = 2z\frac{d^2z}{dx^2} + 2\left(\frac{dz}{dx}\right)^2, \text{we get}$$

$$2x^2z^2\left[2z\frac{d^2z}{dx^2} + 2\left(\frac{dz}{dx}\right)^2\right] + 4z^4 - 4x^2z^2\left(\frac{dz}{dx}\right)^2$$

$$-2xz^2\cdot 2z\frac{dz}{dx} = 0$$

or $\quad x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0$

or $\quad (x^2D^2 - xD + 1)z = 0$

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Unit-II

CHAPTER

4

Differential Equation of First Order but not of the First Degree

DEFINITIONS

1. A differential equation of the form $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots A_{n-1}\, p + A_n = 0$ is called a differential equation of the first and n^{th} degree, where p stands for $\dfrac{dy}{dx}$ and $A_1, A_2, A_3 \dots A_n$ all are functions of x and y.

2. For the equations solvable for p, let the differential equation in above can be resolved into n linear factors such that

$$\{p - f_1(x, y)\}\ \{p - f_2\,(x, y)\}\ \dots\ \{p - f_n\,(x, y)\} = 0$$

Now we can equate each factor to zero and than solve the resulting differential equations of the first order and first degree i.e. the solution is

$$\phi_1(x, y, c_1) = 0,\ \phi_2\,(x, y, c_2) = 0$$
$$\dots\ \phi_n(x, y, c_n) = 0$$

Consider a unique constant C then n solutions are $\phi_1(x, y, c) = 0, \phi_2\,(x, y, c) = 0 \dots \phi_n\,(x, y, c) = 0$

Thus, the combined solution is

$$\phi_1\,(x, y, c)\,\phi_2\,(x, y, c) \dots \phi_n\,(x, y, c) = 0$$

3. For the equations solvable for y let the differential equation is

$$y = f(x,\ p) \qquad \dots(1)$$

Differentiating (1) w.r.t. x we get

$$p = \phi\left(x,\ p,\ \frac{dp}{dx}\right) \qquad \dots(2)$$

where p stands for $\dfrac{dy}{dx}$

Let the solution of (2) is

$$F(x,\ p,\ c) = 0 \qquad \dots(3)$$

Eliminating p between (1) and (3) we get the required solution of (1) say

$$\phi(x,\ y,\ c) = 0$$

If we cannot eliminate p between (1) and (3) then we may solve (1) and (3) such that

$$x = f_1(p, c) \text{ and } y = f_2\,(p, c) \qquad \dots(4)$$

which are the solution of (1).

4. For the equations solvable for x, let the differential equation is

$$x = f\,(y,\ p) \qquad \dots(1)$$

Differentiating (1) w.r.t. y we get

$$\frac{1}{p} = \phi\left(y,\ p,\ \frac{dp}{dy}\right) \qquad \dots(2)$$

Let the solution of (2) is

$$F(y,\ p,\ c) = 0 \qquad \dots(3)$$

Eliminating p between (1) and (3) we get the required solution of (1) such that

$$(x, y, c) = 0$$

If it is not possible to eliminate p, we solve (1) and (3) for x and y such that

$$x = f_1(p,\ c),\ y = f_2(p,\ c) \qquad \dots(4)$$

Which is the required solution of (1).

CLAIRAUT'S EQUATION

1. The differential eqution of the form $y = px + f\,(p)$ is called Clairaut's equation where $f\,(p)$ is a function of p only. Differentiating it w.r.t. x

$$p = p + x\,\frac{dp}{dx} + f\,'(p)\,\frac{dp}{dx}$$

or

$$\frac{dp}{dx}\,\{x + f\,'(p)\} = 0$$

$$\therefore \qquad \frac{dp}{dx} = 0 \quad \text{or} \quad x + f\,'(p) = 0$$

First eduation gives p = constant = c. Thus, we obtained $y = cx + f\,(c)$.

2. If we eliminate p between $y = px + f\,(p)$ and $x + f\,'(p) = 0$ we get the singular solution of given differential equation.

3. Some differentiable equations by suitable change of variables (may be given in questions) may be reduced to Clairaut's form.

4. Let $f\,(x,\ y,\ p) = 0$ be the differential equation of first order and its general solution is

$\phi(x, y, c) = 0$ it represents a curve for each value of c i.e, it represents a family of curves. Hence every differential equation of the first degree represents a family of curves.

SINGULAR SOLUTIONS

1. The singular solution of a differential equation is given by the envelope of family of curves represented by that differential equation.

   Thus, if the envelope of the family of curves

   $$\phi(x, y, c) = 0 \qquad \ldots(1)$$

   represented by the general solution of the differential equation

   $$f(x, y, p) = 0 \qquad \ldots(2)$$

   exists, the equation of the envelope is the singular solution of the differental equation (2).

2. Let the given differential equation be

   $$f(x, y, p) = 0 \qquad \ldots(1)$$

   and its general solution is $\phi(x, y, c) = 0$ ...(2)

   The envelope of the family of curves (2) is contained in the locus obtained by eliminating C, between (2) and

   $$\frac{\partial}{\partial C}\phi(x, y, c) = 0 \qquad \ldots(3)$$

   so the c-eliminant between (2) and (3) be

   $$\psi(x, y) = 0 \qquad \ldots(4)$$

   This is also called c-discriminant relation.

   Similarly, the condition for the equation (1) in p to have two equal roots is obtained on eliminating p between (1) and

   $$\frac{\partial}{\partial p} f(x, y, p) = 0 \qquad \ldots(5)$$

   and the p-eliminant between (1) and (5) is called the p-discriminant relation.

3. If p occurs only in the first degree in the differential equation, there will be no singular solution.

4. The locus common to both the c-discriminant and the p-discriminant relations gives us the singular solutions of (1).

5. Let the general solution of the Clairaut's equation $y = px + f(p)$ ...(1)

   is $y = c(x) + f(c)$ ...(2)

   Differentiating (2) partially w.r.t. c

   $$0 = x + f'(c) \qquad \ldots(3)$$

   The singular solution, which is envelope of (2) is obtained by eliminating c between (2) and (3).

   Again differentiating (1) partially w.r.t. p, we get

   $$0 = x + f'(p) \qquad \ldots(4)$$

   The singular solution of the Clairaut's equation (1) is obtained by eliminating p between (1) and (4).

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The solution of $p^2 - 5p + 6 = 0$ is

   (a) $y - 2x = c$

   (b) $y - 3x = c$

   (c) $(y - 2x - c)(y - 3x - c) = 0$

   (d) $y^2 - 5x + 6 = c$

2. Solution of the equation $y = px + \frac{a}{p}$ is

   (a) $cy = c^2x + a$ (b) $y = x + a$

   (c) $cy = cx + \frac{a}{c}$ (d) $y = cx + a$

3. The solution of the differential equation $y = px + \sqrt{p^2 + 4}$ is

   (a) $(y - cx)^2 = 4cx$ (b) $(y - cx)^2 + 4c^2 = 0$

   (c) $(y - cx)^2 - 4c^2 = 0$ (d) $(y - cx)^2 - c^2 = 4$

4. The singular solution of the differential equation $(xp - y)^2 = p^2 - 1$ is

   (a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$

   (c) $y^2 - x^2 = 1$ (d) $x^2 - 2y^2 = 1$

5. The solution of $p^2 - ax^3 = 0$ is

(a) $25(y + c)^2 - 4ax^5 = 0$

(b) $5(y + C)^2 - 4ax^5 = 0$

(c) $25(y - c)^2 - ax^5 = 0$

(d) $5(y - c)^2 - ax^5 = 0$

6. The solution of the differential equation $xp^2 - (y - x)\,p - y = 0$ is

(a) $(y + x - c)\,(xy - c) = 0$

(b) $(y + 2x + c)\,(xy - c) = 0$

(c) $(2y + x - c)\,(xy - c) = 0$

(d) $(y - x - c)\,(xy - c) = 0$

7. The solution of $y = \sin p - p\cos p$ is

(a) $x = \cos p - c,\ y = \sin p$

(b) $x = c - \cos p,\ y = \sin p - \cos p$

(c) $x = c - \cos p, y = -\cos p$

(d) $x = c - \cos p,\ y = \sin p + \cos p$

8. The solution of the differential equation $p = \log(px - y)$ is

(a) $c = \log(x - y)$ (b) $y = cx + e^{-p}$

(c) $cx = \log(x - y)$ (d) $y = cx - e^{p}$

9. The general solution of $y^2 - 2pxy + p^2(x^2 - 1) = m^2$ is

(a) $(y - cx)^2 = c^2 + m^2$

(b) $y^2 - 2cxy + c^2(x^2 - 1)\,m^2 = 0$

(c) $(y - cx)^2 = c^2$

(d) $y^2 - 2xy + c^2m^2 = 0$

10. The singular solution of $y = px + \frac{a}{p}$ is

(a) $x^2 = 4ay$ (b) $x^2 + y^2 = a^2$

(c) $y^2 = 4ax$ (d) $x^2 - y^2 = a^2$

11. The general solution of $8p^3 - 27y = 0$ is

(a) $y^2 = (x + c)$ (b) $y^2 = (x + c)^2$

(c) $y^2 = 8\,(x + c)^3$ (d) $y^2 = (x + c)^3$

12. The general solution of $(y - px)^2 + a^2p = 0$ is

(a) $(y - x)^2 = a^2c$ (b) $(y - cx)^2 = a^2$

(c) $(y - cx)^2 + a^2c = 0$ (d) $(y - x)^2 + a^2c = 0$

13. The singular solution of $p^2 + y^2 = 1$ is

(a) $y = 1$ only (b) $y = -1$ only

(c) $x = \pm 1$ (d) $y = \pm 1$

14. The differential equation of first degree always represents a

(a) Unique curve (b) Unique surface

(c) Family of curve (d) Family of surfaces

15. The equation $y = px + f(p)$ is called

(a) Linear equation (b) Exact equation

(c) Clairaut's equation (d) Bernoulli's equation

16. The solution of $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$ is

(a) $y = cx$ (b) $y = cx + c$

(c) $y = cX + \frac{1}{c^3}$ (d) $y = cx + c^3$

17. The singular solution of a differential equation contains

(i) Can be obtained from general solution

(ii) Arbitrary constant

(iii)Cannot be obtained from general solution

(iv) Do not contain arbitrary constant

(a) (iii) and (iv) are true (b) (i) and (ii) are true

(c) (i) and (iv) are true (d) (ii) and (iii) are true

18. The solution of $y = 2xp + f(xp^2)$ is

(a) $y = 4cx$ (b) $(y - c^2)^2 = 4cx$

(c) $(y - 4cx)^2 = c^2$ (d) $(y + c)^2 = -4cx$

19. The solution of $\sin px \cos y = p + \cos px \sin y$ is

(a) $y = cx + \sin c$ (b) $y - cx = \sin cx \cos y$

(c) $y = cx - \sin^{-1} c$ (d) $y = cx + \cos^{-1} c$

20. The singular solution of $y^2 - 2pxy + p^2(x^2 - 1) = m^2$ is

(a) $y^2 + m^2x^2 = m^2$ (b) $m^2y^2 + x^2 = m^2$

(c) $y^2 + x^2 = m^2$ (d) $y^2 - x^2 = m^2$

21. The singular solution of $27y - 8p^3 = 0$ is

(a) $27y - 8c^3 = 0$ (b) $27y - 8x^3 = 0$

(c) $x = 0$ (d) $y = 0$

22. The general solution of $y^2 + p^2 = 1$ is

(a) $y^2 + c^2 = 1$ (b) $y = \cos(x + c)$

(c) $y = \sin(x + c)$ (d) None of these

23. The singular solution of $p^2 + px - y = 0$ is

(a) $y^2 = 4x$ (b) $x^2 + 4y = 0$

(c) $x^2 + y^2 = 0$ (d) $y^2 + 4x = 0$

24. The solution of $y = xp + x\sqrt{1+p^2}$ is
(a) $x^2 + y^2 - xc = 0$ (b) $x^3 + y^2 - xc = 0$
(c) $x^3 + y^2 - 2cx = 0$ (d) $x^3 - y^2 - 2cx = 0$

25. The solution of $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$ is
(a) $(y - cx)^2 = b^2 + a^2c^2$
(b) $(y + cx)^2 = b^2 + a^2c^2$
(c) $(y - cx)^2 = b^2 - a^2c^2$
(d) $(y + cx)^2 = b^2 - a^2c^2$

26. The solution of $y = px + ap\,(1 - p)$ is
(a) $y - cx = ac^2$ (b) $y - cx = a - c^2$
(b) $y - cx = ac(1 + c)$ (d) $y - cx = ac(1 - c)$

27. The singular solution of $y = px + \sqrt{a^2p^2 + b^2}$ is
(a) $x^2 + y^2 = a^2 + b^2$ (b) $a^2x^2 + b^2y^2 = 1$
(c) $b^2x^2 + a^2y^2 = a^2b^2$ (d) $a^2x^2 + b^2y^2 = a^2b^2$

28. The singular solution of $(y - px)^2 + pa^2 = 0$ is
(a) $xy = a$ (b) $x - y = \frac{a^2}{4}$
(c) $xy = a^2$ (d) $xy = \frac{a^2}{4}$

29. The envelope of family of curves represented by the differential equation is called
(a) Complete solution (b) Particular integral
(c) Singular solution (d) General solution

30. Singular solution of $p^2 + y^2 = 1$ is
(a) $x^2 + y^2 = 1$ (b) $xy = 1$
(c) $y^2 = 4x$ (d) Not exist

31, The solution of the equation $p^2 + pxy = y^2 \log y$ is
(a) $\log y = x^2 + cX$ (b) $\log x = y^2 + cy$
(c) $c^2 + cxy = y^2 \log y$ (d) $\log y = x^2 + cx + c$

32. The singular solution of $y = px$ is
(a) $y^2 = 4ax$ (b) $x^2 + y^2 = a^2$
(c) $y = mx + c$ (d) Not exist

33. The solution of $(y - px)^2 = p^2(1 - p^2)^2$ is
(a) $y - cx = x^2$ (b) $y = cx + c - c^3$
(c) $y^2 - 2x^2 + 2cx = 0$ (d) $y = cx + c^2$

34. The solution of $(y - px), (p - 1) = p$ is
(a) $(y - x)\,(c - 1) = c$ (b) $(y + x)\,(c - 1) = c$
(c) $(y - cx)\,(c - 1) = c$ (d) $(y - cx)\,c = c^2$

35. The singular solution of $xp^2 - (x - a)^2 = 0$ is
(a) $y = 0$ (b) $xc^2 - (x - a)^2 = 0$
(c) $x - (x - a)^2 = 0$ (d) $x = 0$

36. Which of the following equation is Clairaut's equation
(a) $y = px + f(p)$ (b) $y = p + f(p)$
(c) $y = x + f(p)$ (d) $y = px + f(c)$

37. The solution of $p = xy$ is
(a) $y = ce^x$ (b) $y = ce^{x/2}$
(c) $y = ce^{x/2}$ (d) $y = ce^{x^2}$

38. The order and degree of the differential equation $p^3 + 3p^2 + p - 1 = 0$ is
(a) 3, 1 (b) 1, 3 (c) 3, 3 (d) 1, 1

39. By differentiating w.r.t. x the equation $y = 2px - p^2$, the general solution $x = f(p)$ is
(a) $x = 2p + cp^2$ (b) $x = \frac{2}{3}p + cp^2$
(c) $x = 2p + cp^{-2}$ (d) $x = \frac{2}{3}p + cp^{-2}$

40. The solution of $y^2 + x^2p^2 - 2xyp = 4p^2$ is
(a) $y^2 + x^2c^2 = 4c^2$ (b) $y^2 + x^2c^2 - 2xy = 0$
(c) $(y - cx)^2 = 4/c^2$ (d) $(y - cx)^2 = 2/c$

41. The solution of $(x - a)p^2 + (x - y)p - y = 0$ is
(a) $(y - cx)^2 = 4c$ (b) $y^2 + (x - y)^2 + x - c = 0$
(c) $(x - a)c^2 + (x - y)\,c - y = 0$
(d) $(x - a)^2 + (x - y)\,c = c$

42. The number of constant is a singular solution of n^{th} degree equation is
(a) 0 (b) n (c) $n + 1$ (d) $n - 1$

43. By differentiating w.r.t. y, the equation $xp^3 = a + bp$, The general solution $x = f(p)$ is
(a) $x = ap + \frac{b}{p}$ (b) $x = \frac{a}{p} + \frac{b}{p^2}$
(c) $x = \frac{a}{p^3} + \frac{b}{p^4}$ (d) $x = \frac{a}{p^3} + \frac{b}{p^2}$

44. By differentiating w.r.t. y, the equation $p^3 - 4xyp + 8y^2 = 0$ the general solution $y = f(p)$ is
(a) $p = cy$ (b) $p^2 = cy$
(c) $p^2 + 2p = cy$ (d) None of these

45. Using the transformation $x^2 = u, y^2 = v$, the solution of $x^2(y - px) = yp^2$ is
(a) $y = cx + c^2$ (b) $y^2 = cx + c^2$
(c) $y^2 = cx + c$ (d) $y^2 = cx^2 + c^2$

46. The singular solution of the differential equation $y^2(1+p^2)=a^2$ is

(a) $y^2(1+x^2)=a^2$ (b) $y^2=a^2x$

(c) $y=\pm ax$ (d) $y=\pm a$

47. The singular solution of $p^2(1-x^2)=1-y^2$ is,

(a) $x=\pm 1$ only (b) $y=\pm 1$ only

(c) $x=1, y=1$ (d) $x=\pm 1, y=\pm 1$

48. The solution of $y=2px+y^2p^3$ by substitution $y^2=v$ is

(a) $y^2=cx+\frac{c^3}{8}$ (b) $y^2=cx^2+\frac{c^3}{8}$

(c) $y^2=x+\frac{c}{2}$ (d) $y^2=cx+\frac{c}{2}$

49. By differentiating w.r.t. x, the equation $y=a\sqrt{1+p^2}$ the general solution $x=f(p)$ is

(a) $y=a\sqrt{1+x^2}+p$

(b)$x=a\log\{p+\sqrt{1+p^2}\}+c$

(c) $y=a\log\{p+\sqrt{1+p^2}\}+c$

(d) $y=a\log\{\sqrt{1+p^2}\}+c$

50. The general solution of $y\sqrt{1+p^2}=r$ is

(a) $x^2+y^2=r^2$ (b) $y^2+(x+c)^2=r^2$

(c) $x^2+(y+c)^2=r^2$ (d) $y\sqrt{1+x^2}=r^2$

51. The singular solution of $y=px+a\sqrt{1+p^2}$ is

(a) $y^2=4ax$ (b) $x^2-y^2=1$

(c) $x^2+y^2=a^2$ (d) $x^2=4ay$

52. By differentiating w.r.t. x the equation $y=2px+p^2y$, the general solution $y=f(p)$ is

(a) $p^2y+p+2=0$ (b) $p^2y=c$

(c) $py+p=a$ (d) $py=c$

53. The p-discriminant of $f(x, y, p)=0$ is

(a) $f(x, y, p)=0$ (b) $\frac{\partial}{\partial p}f(x, y, p)=0$

(c) (a) or (b) only (d) Both (a) and (b)

54. The solution of $y=2x\frac{dy}{dx}+3e^{dy/dx}$ is

(a) $y=2cx+3e^c$ (b) $y=2x+3e^x$

(c) $y=2cx+3e^x$ (d) $y=2x+3e^c$

55. One singular solution of $p^3-4pxy+8y^2=0$ is

(a) $y^2=\frac{4}{27}x^3$ (b) $y=\frac{4}{27}x^3$

(c) $y=\frac{27}{4}x^3$ (d) $y^2=\frac{27}{4}x^3$

56. By differentiating w.r.t. x the equation $y=2px+f(xp^2)$, the general solution in the form $x=f(p)$ is

(a) $px=c$ (b) $x\sqrt{p}=c$

(c) $px^{3/2}=c$ (d) $p\sqrt{x}=c$

57. The singular solution of $pxy+x^2=a$ is

(a) $x^2+xy=a$ (b) $x+y^2=a$

(c) $y^2+x=a$ (d) No singular solution

58. The singular solution of $y=px+f(p)$, where p is linear in p is

(a) $y=cx+f(c)$ (b) $y=cx$

(c) $x=cy+f(c)$ (d) No singular solution

59. The degree of the differential equation $p^3+3p^2+2p+x=0$ is

(a) 1 (b) 2 (c) 3 (d) 0

60. The order of the differential equation $p^3-4xyp^2+8y^2=0$ is

(a) 3 (b) 2

(c) 1 (d) None of these

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(a)	**3.**	(d)	**4.**	(b)	**5.**	(a)	**6.**	(d)	**7.**	(b)	**8.**	(d)	**9.**	(a)	**10.**	(c)
11.	(d)	**12.**	(c)	**13.**	(d)	**14.**	(c)	**15.**	(c)	**16.**	(d)	**17.**	(a)	**18.**	(b)	**19.**	(c)	**20.**	(a)
21.	(d)	**22.**	(b)	**23.**	(b)	**24.**	(c)	**25.**	(a)	**26.**	(d)	**27.**	(c)	**28.**	(d)	**29.**	(c)	**30.**	(d)
31.	(a)	**32.**	(d)	**33.**	(b)	**34.**	(c)	**35.**	(d)	**36.**	(a)	**37.**	(c)	**38.**	(b)	**39.**	(d)	**40.**	(c)
41.	(c)	**42.**	(a)	**43.**	(d)	**44.**	(b)	**45.**	(d)	**46.**	(d)	**47.**	(d)	**48.**	(a)	**49.**	(b)	**50.**	(b)
51.	(c)	**52.**	(d)	**53.**	(d)	**54.**	(a)	**55.**	(b)	**56.**	(d)	**57.**	(d)	**58.**	(d)	**59.**	(c)	**60.**	(c)

HINTS AND SOLUTIONS

1. $p^2 - 5p + 6 = 0 \Rightarrow (p-2)(p-3) = 0$

If $p - 2 = 0 \Rightarrow \dfrac{dy}{dx} - 2 = 0 \Rightarrow y - 2x - c = 0$

If $p - 3 = 0 \Rightarrow \dfrac{dy}{dx} - 3 = 0 \Rightarrow y - 3x - c = 0$

Thus, the solution is $(y - 2x - c)(y - 3x - c) = 0$

2. Given equation is $y = px + \dfrac{a}{p}$ which is Clairaut's equation so solution is $y = cx + \dfrac{a}{c} \Rightarrow cy = c^2x + a$

4. Given equation is $(xp - y)^2 = p^2 - 1$

So, $y = px \pm \sqrt{p^2 - 1}$ so its solution is

$$(xc - y)^2 = c^2 - 1$$

i.e. $c^2(x^2 - 1) - 2xyc + y^2 + 1 = 0.$

The envelope of the family of above curve is the required singular solution.

The c-discriminant relation is

$$4x^2y^2 - 4(x^2 - 1)(y^2 + 1) = 0$$

i.e. $x^2 - y^2 = 1$ which is singular solution.

5. Given $p^2 = ax^3 \Rightarrow p = \dfrac{dy}{dx} = a^{1/2}x^{3/2}$ Integrating it

$$y = \frac{2a^{1/2}\,x^{5/2}}{5} + C$$

6. Given that $xp^2 + (y - x)p - y = 0$

i.e. $xp(p-1) + y(p-1) = 0$ or $(p-1)(xp + y) = 0$

If $p - 1 = 0 \Rightarrow \dfrac{dy}{dx} = 1 \Rightarrow y = x + c$ ...(1)

If $xp + y = 0 \Rightarrow p = \dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow \dfrac{dy}{y} + \dfrac{dx}{x} = 0$

Integrating $\log(xy) = \log c \Rightarrow xy = c$ ...(2)

So the solution is $(y - x - c)(xy - c) = 0$

7. Given differential equation is

$$y = \sin p - p\cos p \qquad \text{...(1)}$$

differentiating w.r.t., x, we get

$$p = (\cos p - \cos p + p\sin p)\frac{dp}{dx}$$

or $\quad p = p\sin p\dfrac{dp}{dx} \Rightarrow \sin p\,dp = dx$

Integrating $x = c - \cos p$ ...(2)

(1) and (2) are the required solution.

9. Given that $y^2 - 2pxy + p^2(x^2 - 1) = m^2$

i.e. $(y - px)^2 = p^2 + m^2 \Rightarrow y = px \pm \sqrt{p^2 + m^2}$

which is Clairaut's form, so solution is

$y = cx \pm \sqrt{c^2 + m^2} \Rightarrow (y - cx)^2 = c^2 + m^2$

10. Given differential equation is $y = px + \dfrac{a}{p}$

i.e. $xp^2 - yp + a = 0$

its discriminant is $y^2 - 4ax = 0$

So singular solution is $y^2 = 4ax$

13. Given that $p^2 + y^2 = 1 \Rightarrow \dfrac{dy}{dx} = \pm\sqrt{1 - y^2}$

or $\quad dx = \pm\dfrac{dy}{\sqrt{1 - y^2}}$ integrating it,

we get, $x + c = \pm\cos^{-1} y$

$\Rightarrow y = \cos(x + c)$ which is general solution.

Again p-discriminant is $y^2 - 1 = 0 \Rightarrow y = \pm 1$ which satisfy the given differential equation, so it is singular solution.

18. Given that $y = 2px + f(xp^2)$ ...(1)

Differentiating w.r.t. $x\,p = 2p + 2x\dfrac{dp}{dx}$

$$+ f'(xp^2)\left\{p^2 + 2xp\frac{dp}{dx}\right\}$$

or $\quad \left(p + 2x\dfrac{dp}{dx}\right)[1 + pf'(xp^2)] = 0$

If $p + 2x\dfrac{dp}{dx} = 0 \Rightarrow \dfrac{dp}{p} = \dfrac{-dx}{2x}$ integrating it

$\log p = \dfrac{-1}{2}\log x + \log c \Rightarrow px^{1/2} = c$

$\Rightarrow p = \dfrac{c}{\sqrt{x}}$ put it in (1) we get $y = 2c\sqrt{x} + f(c^2)$

19. Given $\sin px\cos y - \cos px\sin y = p$ i.e. $\sin(-y + px) = p$

$\Rightarrow y = px - \sin^{-1} p$ which is Clairaut's equation so its solution is

$$y = cx - \sin^{-1} c$$

21. Given $27y - 8p^3 = 0 \Rightarrow p = \dfrac{dy}{dx} = \dfrac{3}{2}y^{1/3}$

$\Rightarrow dy\left(\dfrac{2}{3}y^{-1/3}\right) = dx$ Integrating it $y^{2/3} = x + c$

$\Rightarrow y^2 = (x+c)^3$ which is general solution.

Differentiating it w.r.t. c $x + c = 0$, put $x = -c$ in (1) we get $y = 0$ so $p = 0$ i.e. singular solution is $y = 0$.

23. Given that $p^2 + px - y = 0$ its p-discriminant is $x^2 + 4y = 0$ which is singular solution.

27. Given differential equation is $y = px + \sqrt{a^2p^2 + b^2}$

or $$(y - px)^2 = a^2p^2 + b^2$$

$$\Rightarrow \quad (x^2 - a^2)p^2 - 2xyp + y^2 - b^2 = 0$$

So p-discriminant is

$$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$ which is singular solution.

31. Given differential equation is $y^2 \log y = pxy + p^2$

or $x = \dfrac{y\log y}{p} - \dfrac{p}{y}$ differentiating w.r.t. y

$$\frac{1}{p} = \frac{\log y}{p} + \frac{y}{yp} - \frac{y\log y}{p^2}\frac{dp}{dy} + \frac{p}{y^2} - \frac{dp}{dy} - \frac{1}{y}$$

or $$\frac{p}{y^2}\left(1 + \frac{y^2}{p^2}\log y\right) = \frac{1}{y}\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right)$$

or $\dfrac{p}{y^2} = \dfrac{1}{y}\dfrac{dp}{dx}$ i.e. $\dfrac{dp}{p} = \dfrac{dy}{y}$ integrating it

$\log p = \log y + \log c$ or $p = cy$,
put it in original equation we get required solution
$\log y = cx + c^2$

34. $y = px = \dfrac{p}{p-1}$ which is Clairaut's form so its solution is $y = cx + \dfrac{c}{c-1}$

35. Given that $xp^2 - (x-a)^2 = 0$

$$\Rightarrow \quad p = \frac{dy}{dx} = \pm(x^{1/2} - ax^{-1/2})$$

i.e. $$dy = \pm(x^{1/2} - ax^{-1/2})dx$$ integrating it

$$y + c = \pm\left(\frac{2}{3}x^{3/2} - 2ax^{1/2}\right)$$

$$\Rightarrow \quad 9(y+c)^2 = 4x(x-3a)^2 \qquad \ldots(1)$$

Differentiating at w.r.t. c. we get $y + c = 0$ …(2)

Also $p-$ discriminant is $x(x-a)^2 = 0$ so singular solution is $x = 0$

39. Given $y = 2px - p^2$ differentiate w.r.t. x we get

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} \Rightarrow p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} = 0$$

or $$\frac{dx}{dp} = \frac{2(p-x)}{p} \Rightarrow \frac{dx}{dp} + \left(\frac{2}{p}\right)x = 2$$

its I.F. is $e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$ so its solution is

$$xp^2 = \int p^2.2dp + c = \frac{2}{3}p^3 + c$$

i.e. $$x = \frac{2}{3}p + \frac{c}{p^2}$$

41. Given $(x-a)p^2 + (x-y)p - y = 0$

$$\Rightarrow \quad xp^2 - ap^2 + xp - yp - y = 0$$

or $$p(-y + xp) - (y - xp) = ap^2$$

$$\Rightarrow \quad (y - px)(-1 - p) = ap^2$$

or $y = px - \dfrac{ap^2}{1+p}$ which is Clairant's form so its solution is $y = cx - \dfrac{ac^2}{1+c}$

44. Given $p^3 - 4xyp + 8y^2 = 0$

$x = \dfrac{p^2}{4x} + \dfrac{2y}{p}$ differentiating w.r.t. y, we get

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

or $$\frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = \frac{p^2}{4y^2} - \frac{1}{p} = \frac{p}{2y}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$$

So, $\dfrac{dp}{dy} = \dfrac{p}{2y} \Rightarrow \dfrac{2dp}{p} = \dfrac{dy}{y}$ integrating, we get

$\log p^2 = \log cy \Rightarrow p^2 = cy$

45. Given equation is $x^2(y - px) = yp^2$ and $x^2 = u$, $y^2 = v$

i.e. $2xdx = du$ and $2y\,dy = dv$ so $\dfrac{dy}{dx} = p = \dfrac{x}{y}\dfrac{dv}{du}$

put it in original equation, we get

$$x^2\left(y - x\cdot\frac{x}{y}\frac{dv}{du}\right) = y\,\cdot.\,\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2$$

or $$y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2 \Rightarrow v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

i.e. $v = u\dfrac{dv}{du} + \left(\dfrac{dv}{du}\right)^2 \Rightarrow v = uP + P^2$, when $\dfrac{dv}{du} = P$

This is Clairaut's form so its solution is $v = uc + c^2$

$\Rightarrow y^2 = cx^2 + c^2$

48. Given $y = 2px + y^2p^3$ and $y^2 = v$

i.e. $2y\dfrac{dy}{dx} = \dfrac{dv}{dx} \Rightarrow 2py = P$, where $P = \dfrac{dv}{dx}$ or $p = \dfrac{P}{2y}$

put it in above equation, $y = 2x\dfrac{P}{2y} + y^2\left(\dfrac{P^3}{8y^3}\right)$

$\Rightarrow y^2 = xP + \dfrac{P^3}{8}$ or $v = xP + \dfrac{P^3}{8}$ which is in Clairaut's form so its solution is $y^2 = cx + \dfrac{c^3}{8}$.

55. Given that $p^3 - 4pxy + 8y^2 = 0$ ...(1)

Differentiate w.r.t. x $3p^2 - 4xy = 0$

i.e. $p = 2\sqrt{\dfrac{xy}{3}}$ ...(2)

Eliminate p between (1) and (2) we get $27y - 4x^3$.

Also $y = \dfrac{4}{27}x^3$ satisfy the given differential equation.

57. $pxy + x^2 = a$ is linear in p so no singular solution exist.

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Unit-II

CHAPTER 5

Orthogonal Trajectories

DEFINITIONS

1. A trajectory of a given system of curves is defined to be a curve which cuts all the members of the family according to a given law. If the curve cuts at constant angle α, it is called an α-trajectory.

   If the angle is at right angle, the trajectories are called orthogonal trajectories.

   If the angle is other than 90° the trajectories are called oblique trajectories.

2. In cartesian co-ordinates, the equation of the orthogonal trajectories can be obtained as follows :

(i) Differentiate the equation of the given family of curves and eliminate the arbitrary parameter. Thus, the differential equation of given family of curves is obtained.

(ii) In this differential equation, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and we obtained the differential equation of the orthogonal trajectories is obtained.

(iii) Find the general solution of this differential equation. This is the required equation of orthogonal trajectories.

3. In polar co-ordinates, let the equation of given family of curver be $f(r, \theta, c) = 0$ ...(1)

   Differentiating it w.r.t. θ and eliminating c, the differential equation of the given family (1) be

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \qquad \text{...(2)}$$

   The differential equation of the required trajectories is

$$\phi\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$$

   Solving this we shall get the required equation of the orthogonal trajectories.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The orthogonal trajectories of the family of curves $y = ax^2$ is

   (a) $x^2 + y^2 = c^2$ (b) $x^2 + 3y^2 = c^2$

   (c) $x^2 + 2y^2 = c^2$ (d) $x^2 - 2y^2 = c^2$

2. The orthogonal trajectories of the family of curves $ay^2 = x^3$ is

   (a) $x^2 - \frac{3}{2}y^2 = c^2$ (b) $x^2 + 3y^2 = c^2$

   (c) $x^2 + \frac{3}{2}y^2 = c^2$ (d) $x^2 - 3y^2 = c^2$

3. The orthogonal trajectories of the family of circles $(x-1)^2 + y^2 + 2ax = 0$, where a is parameter

   (a) $x^2 + y^2 = cy + 1$ (b) $x^2 + y^2 = cy$

   (c) $x^2 - y^2 = cy$ (d) $x^2 + y^2 = cy^2$

4. The orthogonal trajectories of the cardioids $r = a(1 + \cos\theta)$; where a is the parameter is

   (a) $r = b\cos\theta$ (b) $r = b(1 - \cos\theta)$

   (c) $r = b\cos^2\theta + 1$ (d) $r = b(1 + \cos\theta)$

5. The differential equation of family of orthogonal trajectories of the family of circles $x^2 + y^2 = 2ax$ is

   (a) $y' = \frac{x^2 - y^2}{xy}$ (b) $y' = \frac{x^2 + y^2}{2xy}$

   (c) $y' = \frac{x^2 - y^2}{2xy}$ (d) $y' = \frac{x^2 + y^2}{xy}$

6. The orthogonal trajectories of the family of curves $y = ax^4$ is

(a) $x^2 + ny^2 = c^2$ (b) $x^2 + y^2 = n^2$

(c) $nx^2 + y^2 = c^2$ (d) $x^2y^2 = n^2c^2$

7. The orthogonal trajectories of the $r\theta = a$

(a) $r = ae^{\theta}$ (b) $r = ae^{\theta^2}$

(c) $r^2 = ae^{\theta}$ (d) $r^2 = ae^{\theta^2}$

8. The orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$, where a is parameter is

(a) $xy = c$ (b) $x^2y = c$

(c) $x^2 = cy$ (d) $x = cy$

9. The differential equation of orthogonal trajectories of the family of curve $y^2 = 4a(x + a)$ is

(a) $yy'^2 + 2xy' = y$ (b) $xy'^2 + xy' = y$

(c) $yy'^2 + y' = x$ (d) $yy'^2 + xy' = 2y$

10. The orthogonal trajectories of the family of rectangular hyperboloid $xy = c^2$ is

(a) $x + y = k$ (b) $x - y = k$

(c) $x^2 + y^2 = k$ (d) $x^2 - y^2 = k$

11. The orthogonal trajectories of $r = e^{a\theta}$ is

(a) $\theta + \log r = k$ (b) $\theta^2 + (\log r)^2 = k$

(c) $\theta^2 + \log r = k$ (d) $\theta + (\log r)^2 = k$

12. The orthogonal trajectories of $r^n \sin n\theta = a^n$ is

(a) $r^n \operatorname{cosec} n\theta = b^n$ (b) $r^n \sec n\theta = b^n$

(c) $r^n \cos n\theta = b^n$ (d) $r^n \tan n\theta = b^n$

13. The differential equation of family of curves of $y^2 - x^2 + 4xy - 2cx = 0$ is

(a) $2x(2x + y)y' = x^2 + y^2$

(b) $(x + 2y)y' = x^2 + y^2$

(c) $2x(x + y)y' = x^2 + y^2$

(d) $(2x + y)y' = x^2 + y^2$

14. The orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2gx + c = 0$, g is parameter is

(a) $x^2 + y^2 + a^2$ (b) $x^2 + y^2 + x + c = 0$

(c) $x^2 + y^2 + 2gy + c = 0$ (d) $x^2 + y^2 + 2y + c = 0$

15. The orthogonal trajectories of the family of circles $x^2 + y^2 = 2ax$, a is parameter is

(a) $x^2 = 4ay$ (b) $x^2 + y^2 = bx$

(c) $y^2 = 4ax$ (d) $x^2 + y^2 = by$

16. The orthogonal trajectories of $r^n = a^n \cos n\theta$, a is a parameter is

(a) $r^n = c^n \cos n\theta$ (b) $r^n \sin n\theta = c^n$

(c) $r^n \cos n\theta = c^n$ (d) $r^n = c^n \sin n\theta$

17. The differential equation of the family of circles $x^2 + y^2 + 2fy + 1 = 0$, where f is parameter is

(a) $x^2 + y^2 + 2xy' = 0$

(b) $x^2 - y^2 - 2xyx' + 1 = 0$

(c) $x^2 - y^2 + yx' + 1 = 0$

(d) $x^2 + y^2 + yx' - 1 = 0$

18. The differential equation of the orthogonal trajectories of the curve $r = e^{a\theta}$ is

(a) $\frac{d\theta}{dr} = -\frac{\log r}{r\theta}$ (b) $\frac{d\theta}{dr} = -\frac{r\theta}{\log r}$

(c) $\frac{d\theta}{dr} = \frac{\log r}{r\theta}$ (d) $\frac{d\theta}{dr} = \frac{r\theta}{\log r}$

19. The differential equation of the orthogonal trajectories of the family of cardioids $r = a(1 + \cos\theta)$ is

(a) $\frac{dr}{d\theta} = \frac{1 + \cos\theta}{\sin\theta}$ (b) $\frac{dr}{d\theta} = \frac{r\sin\theta}{1 + \cos\theta}$

(c) $\frac{dr}{d\theta} = \frac{r(1 + \cos\theta)}{\sin\theta}$ (d) $\frac{dr}{d\theta} = -\frac{r\sin\theta}{1 + \cos\theta}$

20. The equation of the family of curves that is orthogonal to $ax^2 + y^2 = 1$ is

(a) $y = ce^{x+y}$ (b) $y^2 = ce^{x^2+y^2}$

(c) $y^2 = ce^{x^2+y}$ (d) $y^2 = ce^{x+y^2}$

21. The differential equation of the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$, where a is parameter is

(a) $xdx = ydy$ (b) $x^{2/3}dx = y^{2/3}dy$

(c) $x^{1/3}dx = y^{1/3}dy$ (d) None of these

22. The orthogonal trajectories of the curve $x^2 + y^2 = a^2$, where a is the parameter is

(a) $xy = c$ (b) $\frac{x}{y} = c$

(c) $x^2y = c$ (d) $\frac{y^2}{x} = c$

23. To find the orthogonal trajectories in polar curve, $r\frac{d\theta}{dr}$ is replaced by

(a) $r\frac{dr}{d\theta}$ (b) $\frac{1}{r}\frac{dr}{d\theta}$ (c) $-r\frac{dr}{d\theta}$ (d) $-\frac{1}{r}\frac{dr}{d\theta}$

24. The differential equation of the orthogonal trajectories we replace $\frac{dy}{dx}$ by

(a) $\frac{dx}{dy}$ (b) $-\frac{dx}{dy}$ (c) $-x\frac{dx}{dy}$ (d) $-y\frac{dx}{dy}$

25. The differential equation of orthogonal trajectories of the curve $xy = a^2$ is given by

(a) $dx = ydy$ (b) $xdx = dy$
(c) $ydx = xdy$ (d) $xdx = ydy$

26. A curve which cuts every member of a given family of curves according to some given law is called a

(a) Orthogonal trajectories
(b) Trajectories
(c) Self orthogonal families
(d) Orthogonal curves

27. The orthogonal trajectories of a system of concurrent straight lines is a system of

(a) Concentric circles (b) Parabolas
(c) Ellipses (d) Itself

28. The orthogonal trajectories of the family of parabolas $y^2 = a(x + a)$ where a is a parameter a family of

(a) Liner (b) Ellipses
(c) Parabolas (d) Hyperbolas.

29. The equation of the system of orthogonal trajectories of the parabolas $r = \frac{2a}{1+\cos\theta}$, a is parameter is

(a) $r = 2b(1+\cos\theta)$ (b) $r = 2b\sin\theta$
(c) $r = 2b\cos\theta$ (d) $\frac{2b}{1-\cos\theta}$

30. The orthogonal trajectories of the family of curves $r^2 = a^2\cos 2\theta$ is

(a) $r^2 = b^2\sin 2\theta$ (b) $r^2 = \frac{b^2}{\sin^2\theta}$
(c) $r^2 = b^2\tan^2\theta$ (d) $r^2 = b^2\cot 2\theta$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(c)	**3.**	(a)	**4.**	(b)	**5.**	(c)	**6.**	(a)	**7.**	(d)	**8.**	(d)	**9.**	(a)	**10.**	(d)
11.	(b)	**12.**	(c)	**13.**	(a)	**14.**	(c)	**15.**	(d)	**16.**	(d)	**17.**	(b)	**18.**	(a)	**19.**	(c)	**20.**	(b)
21.	(c)	**22.**	(b)	**23.**	(d)	**24.**	(b)	**25.**	(d)	**26.**	(b)	**27.**	(a)	**28.**	(c)	**29.**	(d)	**30.**	(a)

HINTS AND SOLUTIONS

1. Given $y = ax^2$, differentiating it, $\frac{dy}{dx} = 2ax$

eliminate a between them we get

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$ this is the differential equation of the given family of curves so differential equation of the orthogonal trajectories is $\frac{1}{y}\left(-\frac{dx}{dy}\right) = \frac{2}{x}$ i.e.

$xdx + 2ydy = 0$

integrating it, $\frac{x^2}{2} + y^2 = \frac{1}{2}c^2 \Rightarrow x^2 + 2y^2 = c$

This is the required orthogonal trajectories.

3. Given curve is $x^2 - 2x + 1 + y^2 + 2ax = 0$ differentiate it $2x - 2 + 2y\frac{dy}{dx} + 2a = 0$ Thus, the differential equation of family of curves is

$$x^2 - 2x + 1 + y^2 + \left(2 - 2x - 2y\frac{dy}{dx}\right)x = 0$$

$$y^2 - x^2 - 2xy\frac{dy}{dx} + 1 = 0$$

So the differential equation of orthogonal trajectories is

$$y^2 - x^2 + 2xy\frac{dx}{dy} + 1 = 0$$

or $2xydx + (y^2 - x^2 + 1)dy = 0 = Mdx + Ndy$

$$\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = -2x$$

So, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{2xy}(-4x) = -\frac{2}{y}$

So integrating factor is $e^{-\int 2/y\,dy} = \frac{1}{y^2}$

Thus $\frac{2x}{y}dx + \left(1 + \frac{1-x^2}{y^2}dy\right) = 0$ is *exact* so integrating it we get

$$\frac{x^2}{y} + y - \frac{1}{y} = c \Rightarrow x^2 + y^2 - 1 = cy$$

4. Givn curve is $r = a(1+\cos\theta), \frac{dr}{d\theta} = -a\sin\theta$

eliminate a between two equation

$$r = -\frac{(1+\cos\theta)}{\sin\theta}\frac{dr}{d\theta}$$

This is the differential equation of family of curves. So differential equation of family of orthogonal trajectories is

$$r = \frac{-(1+\cos\theta)}{\sin\theta}\left(-r^2\frac{d\theta}{dr}\right)$$

so $$1 = \left(\frac{1+\cos\theta}{\sin\theta}\right)\frac{rd\theta}{dr}$$

i.e. $$\frac{dr}{r} - \frac{1+\cos\theta}{\sin\theta}d\theta = 0$$

Integrating it $\int\frac{dr}{r} - \int\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}d\theta = \log b$

$\log r - 2\log\sin\theta/2 = \log b$

$$r = b\sin^2\frac{\theta}{2} = \frac{b}{2}(1-\cos\theta) = c(1-\cos\theta)$$

6. Given $y = ax^n$, differentiate it w.r.t. x we get $\frac{dy}{dx} = a_n x^{n-1}$, eliminate a between them we get the differential equation of family of curves *i.e.* $x\frac{dx}{dy} = ny$, so differential equation of orthogonal trajectories is $x\left(-\frac{dy}{dx}\right) = ny \Rightarrow -xdx + nydy = 0$

Integrating we get $x^2 + ny^2 = c^2$

9. Given curve is $y^2 = 4a(x+a)$

Differentiate it $2y\frac{dy}{dx} = 4a$, eliminate a between them we get the differential equation of given family of curves *i.e.*

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

So the differential equation of family of orthogonal trajectories is

$$y = -2\frac{dx}{dy}\left(x - \frac{y}{2}\frac{dx}{dy}\right)$$

i.e. $$y = -\frac{2}{y'}\left(x - \frac{y}{2y'}\right) = \frac{-2(2xy' - y)}{2y'^2}$$

$\Rightarrow \quad yy'^2 + 2xy' = y$

10. Given curve is $xy = c^2$, differentiate it w.r.t. x, $\frac{xdy}{dx} + y = 0$, this is the differential equation of the given family of curves. So differential equation of the orthogonal trajectories is $x\left(-\frac{dx}{dy}\right) + y = 0$

$\Rightarrow xdx = ydy$ integrating it $x^2 - y^2 = c$

12. Given family of curve is $r^n\sin n\theta = a^n$

Differentiate it w.r.t. θ we get

$$nr^n\cos n\theta + nr^{n-1}\sin n\theta\frac{dr}{d\theta} = 0$$

i.e., $\frac{dr}{d\theta} = -r\cot n\theta$ which is the differential equation of the given family of curves.

So the differential equation of the orthogonal trajectories is $-r^2\frac{d\theta}{dr} = -r\cot n\theta$

i.e. $$\frac{1}{r}dr = \tan n\theta d\theta$$

Integrating it, $\log r = -\frac{1}{n}\log(\cos n\theta) + \log c$

$$n\log r = -\log\cos n\theta + n\log c$$

$\Rightarrow \quad r^n = c^n/\cos n\theta \Rightarrow r^n\cos n\theta = c^n$

15. Given family of curve is $x^2 + y^2 = 2ax$

differentiate it w.r.t. x, $2x + 2y\frac{dy}{dx} = 2a$

i.e., $x + y\frac{dy}{dx} - a = 0$, eliminate a between two equations $x^2 + 2xy\frac{dy}{dx} - y^2 = 0$ which is the differential equation of family of curves. So differential equation of orthogonal trajectories is $x^2 + 2xy\left(-\frac{dx}{dy}\right) - y^2 = 0 \Rightarrow \frac{dx}{dy} = \frac{x^2 - y^2}{2xy}$

put $\quad x = vy, \frac{dx}{dy} = v + \frac{ydv}{dy}$ we get,

$\frac{ydv}{dy} = \frac{v^2 - 1}{2v} - v \Rightarrow \frac{2vdv}{1+v^2} = -\frac{dy}{y}$, integrating it $\log(1+v^2) = -\log y + \log b \Rightarrow 1 + \frac{x^2}{y^2} = \frac{b}{y}$

$\therefore \quad x^2 + y^2 = by$

17. Given family of curves is $x^2 + y^2 + 2fy + 1 = 0$

Differentiate w.r.t. x, $2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0$

i.e., $x\frac{dx}{dy}+y+f=0 \Rightarrow f=-y-x\frac{dx}{dy}$

eliminate f between two equations we get

$$x^2-y^2-2xy\frac{dx}{dy}+1=0$$

18. Given $r=e^{a\theta}$ differentiate w.r.t. θ we get

$$\frac{dr}{d\theta}=ae^{a\theta}=ar$$

Eliminate a between two equations we get

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\log r}{\theta}$$

Which is the differential equation of given family of curves. So differential equation of orthogonal trajectories is

$$-\frac{r d\theta}{dr}=\frac{\log r}{\theta}\Rightarrow\frac{d\theta}{dr}=-\frac{\log r}{r\theta}$$

20. Given that $ax^2+y^2=1$, differentiate it w.r.t. x we get $\frac{dy}{dx}=-\frac{ax}{y}$, eliminate a between them we get $\frac{dy}{dx}=\frac{y^2-1}{xy}$. So differential equation of orthogonal trajectories is $-\frac{dx}{dy}=\frac{y^2-1}{xy}$ i.e., $xdx=\frac{1-y^2}{y}dy$, integrating we get

$$\frac{1}{2}x^2=\log y-\frac{1}{2}y^2-\frac{1}{2}\log c$$

$$\Rightarrow \qquad x^2=\log y^2-y^2-\log c$$

$$\log(y^2/c)=x^2+y^2 \text{ or } y^2=ce^{x^2+y^2}$$

22. Given that $x^2+y^2=a^2$, differentiate it w.r.t. x we get $\frac{dy}{dx}=-\frac{x}{y}$, it is the differential equation of the given family of curves. So differential equation of the orthogonal trajectories is

$$-\frac{dx}{dy}=-\frac{x}{y}\Rightarrow\frac{dx}{x}=\frac{dy}{y}$$

Integrating it, $\log x=\log y+\log c\Rightarrow x=cy$

25. Given that $xy=a^2$

differentiate w.r.t. x, we get

$\frac{xdy}{dx}+y=0$ so differential equation of the family of orthogonal trajectories is

$x\left(-\frac{dx}{dy}\right)+y=0\Rightarrow xdx=ydy$

29. $r=\frac{2a}{1+\cos\theta}$, differentiating w.r.t. θ, $\frac{dr}{d\theta}=\frac{2a\sin\theta}{(1+\cos\theta)^2}$

eliminate a between two equations, we get

$\frac{dr}{d\theta}=\frac{r\sin\theta}{1+\cos\theta}$, so the differential equation of orthogonal trajectories is $-\frac{r^2d\theta}{dr}=\frac{r\sin\theta}{1+\cos\theta}$

i.e. $\frac{dr}{r}=-\frac{1+\cos\theta}{\sin\theta}d\theta=-\cot\frac{\theta}{2}d\theta$

integrating we get, $\log r=-2\log\sin\frac{\theta}{2}+\log c$

or $\log\frac{r}{c}=\log\frac{1}{\sin^2\theta/2}\Rightarrow r=\frac{c}{\sin^2\theta/2}=\frac{2c}{1-\cos\theta}$

* * *

Unit-II

CHAPTER

6

Ordinary Simultaneous Differential Equations

INTRODUCTION

1. In this chapter, we shall discuss ordinary differential equations containing one independent variable and two or more then two dependent variables.

   Let x and y be two dependent variables and t be the independent variable. Then the simultaneous linear differential equations are

   $$f_1(D)x + f_2(D)y = F_1(t) \quad ...(1)$$

   and $$\phi_1(D)\,x + \phi_2(D)\,y = F_2(t) \quad ...(2)$$

   where $F_1(t)$ and $F_2(t)$ are functions of t.

2. There are two methods to solve equations (1) and (2) first method is method of elimination and second method is method of differentiation.

3. **Method of Elimination :** Operating both sides of (1) by $\phi(D)$ and of (2) by $f_2(D)$ and then subtracting, we get

   $$[\phi_2(D)f_1(D) - f_2(D)\,\phi_1(D)]\,x$$
   $$= \phi_2(D)\,F_1(t) - f_2(D)\,F_2(t)$$

   i.e. $$F(D)x = F(t)$$

   which is linear differential equation with constant coefficients in x and t. After solving it we find x, then put this x in (1) or (2) we get value of y.

4. **Method of Differentiation :** Let the two given equation (1) and (2) connect four quantities $x, y, \frac{dx}{dt}$ and $\frac{dy}{dt}$. Differentiating (1) and (2) w.r.t. t we have four equations containing $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$. Eliminate $y, \frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ from these four equations we get a differential equations of order two in x and t. Solving it we get x in terms of t. Then by other equations we get the variable y.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. In equations $\frac{dx}{dt} = -\omega y, \frac{dy}{dt} = \omega x$, the value of x is

   (a) $e^{\omega t}(A\cos\omega t + B\sin\omega t)$

   (b) $A\cos\omega t + B\sin\omega t$

   (c) $e^{-\omega t}(A\cos\omega t + B\sin\omega t)$

   (d) $e^{\omega t}(A\omega t + B)$

2. Solution for x of the following equations $(D-7)x + y = 0$ and $-2x + (D-5)\,y = 0$ is

   (a) $e^{6t}(A\cos t + B\sin t)$ (b) $e^{4t}(A\cos t - B\sin t)$

   (c) $e^{-6t}(A\cos t - B\sin t)$ (d) $e^{-4t}(A\cos t + B\sin t)$

3. If $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} = t^2 - 5$ then x is

   (a) $A\cos t + B\sin t + t$ (b) $A\cos t - B\sin t + t^2 + 1$

   (c) $A\cos t + B\sin t + t^2 - 1$

   (d) None of these

4. If $\frac{dx}{dt} = y + 1$ and $\frac{dy}{dt} = x + 1$ then x is

   (a) $Ae^{t} + Be^{-t}$ (b) $Ae^{2t} + Be^{-t}$

   (c) $Ae^{t} + Be^{-2t}$ (d) $Ae^{t} + Be^{-t} - 1$

5. If $Dx = 2y$, $Dy = 2z$, $Dz = 2x$ then the differential equation for x is

   (a) $(D^3 - 8)x = 0$ (b) $(D^3 - 2D)x = 0$

   (c) $(D^3 - 4D + 2)x = 0$ (d) $(D^3 + 8)x = 0$

6. If $D^2y = x$ and $D^2x = y$ then the solution for y is

   (a) $Ae^{t} + Be^{-t} + C\cos t - D\sin t$

(b) $Ae^{t}+Be^{-t}-C\cos 2t-D\sin 2t$

(c) $Ae^{2t}+Be^{-2t}-C\cos t-D\sin t$

(d) $Ae^{2t}+Be^{-2t}-C\cos 2t-D\sin 2t$

7. The solution of $\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$ is

(a) Unique (b) Finite (c) Infinite (d) Not exist

8. If $\frac{dx}{dt}+4y=0$ and $-4x+\frac{dy}{dt}=0$ then x is

(a) $A\sin 4t+B\cos 4t$ (b) $A\sin 4t-B\cos 4t$

(c) $-A\sin 4t+B\cos 4t$ (d) $-A\sin 4t-B\cos 4t$

9. If $D^2x+y=\sin t$ and $D^2y+x=\cos t$ then the differential equation containing x is

(a) $(D^4-1)x=-\sin t-\cos t$

(b) $(D^4-1)x=\sin t+\cos t$

(c) $(D^4-1)x=\sin t-\cos t$

(d) $(D^4-1)x=-\sin t+\cos t$

10. If $\frac{dx}{dt}+5y=0$ and $\frac{dy}{dt}-5x=0$ then y is

(a) $A\cosh 5t+B\sinh 5t$ (b) $Ae^{5t}+Be^{-5t}$

(c) $(A+Bt)e^{5t}$ (d) $A\cos 5t+B\sin 5t$

11. If $Dx+Dy+2x+y=0$ and $Dy+5x+3y=0$ then x is

(a) $A\cosh t+B\sinh t$ (b) $Ae^{t}+Be^{-t}$

(c) $A\cos t+B\sin t$ (d) $(A+Bt)e^{t}$

12. If $Dx=ny-mz, Dy=lz-nx, Dz=mx-ly$ then the differential equation for x is

(a) $(D^2-l^2-m^2-n^2)x=lc$

(b) $(D^2+l^2+m^2+n^2)x=lc$

(c) $(D^2+l^2+m^2+n^2)x=(l+m)c$

(d) $(D^2-l^2-m^2-n^2)x=(l+m+n)c$

13. If $x\frac{dy}{dx}+z=0$ and $\frac{xdz}{dx}+y=0$ then the differential equation for y is

(a) $(D^2-1)y=0$ (b) $(D^2+1)y=0$

(c) $(D^2-2D+1)y=0$ (d) None of these

14. Is $t\,dx=(t-2x)\,dt$ and $t\,dy=(tx+ty+2x-t)$ then the solution for x is

(a) $t+\frac{A}{t}$ (b) $\frac{t}{2}+\frac{A}{t^2}$ (c) $\frac{t}{4}+\frac{A}{t}$ (d) $\frac{t}{3}+\frac{A}{t^2}$

15. In equation $\frac{dx}{dt}=-\omega y$, $\frac{dy}{dt}=\omega x$ the value of y is

(a) $A\sin\omega t-B\cos\omega t$ (b) $e^{\omega t}(A\sin\omega t+B\cos\omega t)$

(c) $e^{\omega t}(A\cos\omega t-B\sin\omega t)$ (d) $e^{\omega t}(A\cos\omega t+B\sin\omega t)$

16. If $(D-3)x+8y=0$ and $x+(D+3)y=0$ then x is

(a) $Ae^{2t}+Be^{-2t}$ (b) $x=Ae^{t}+Be^{-t}$

(c) $Ae^{t}+Be^{-2t}$ (d) $Ae^{2t}+Be^{-t}$

17. If $(D^2-3)x-4y=0$ and $x+(D^2+1)y=0$ then the differential equation for y is

(a) $(D^2+1)y=0$ (b) $(D^2+1)^2y=0$

(c) $(D^2-1)^2y=0$ (d) $(D^2-1)y=0$

18. The locus of the point satisfying the equation $Dx=-\omega y$ and $Dy=+\omega x$ is

(a) Line (b) Parabola

(c) Ellipse (d) Circle

19. The solution of $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ is

(a) $x=c_1 2, y=c_2 z$ (b) $x=c_1 y, y=c_2 z^2$

(c) $x=c_1 z^2, y=c_2 z$ (d) $x=c_1 y^2, y=c_2 z$

20. The solution of $\frac{xdx}{y^2z}=\frac{dy}{xz}=\frac{dz}{y^2}$ is

(a) $x^2-y^2=c_1,\ x^2-z^2=c_2$

(b) $x^3-y^3=c_1,\ x^2-z^2=c_2$

(c) $x^3+y^3=c_1,\ x^2+z^2=c_2$

(d) $x^2+y^2=c_1,\ x^2+z^2=c_2$

21. The solution of $\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$ is

(a) $x^2+y^2+z^2=c_1, x+y+z=c_2$

(b) $x^2y^2z^2=c_1, x+y+z=c_2$

(c) $x^2+y^2+z^2=c_1, xyz=c_2$

(d) $x+y-z=c_1, xyz=c_2$

22. If $Dx+y=\sin t$, $Dy+x=\cos t$ with $x(0)=0$, $x'(0)=1$ then x is

(a) $2e^{t}+3e^{-t}$ (b) $\frac{1}{2}e^{t}+e^{-t}$

(c) $2e^{t}+\frac{1}{3}e^{-t}$ (d) $e^{t}-e^{-t}$

23. The solution of $\frac{dx}{y^2z}=\frac{dy}{-x^2z}=\frac{dz}{x^2y}$ is

(a) $f(x^3+y^3, y^2+z^2)=0$

(b) $f(x^3+y^3+z^3, y^2+z^2)=0$

(c) $f(x^3+y^3+z^3, x^2+y^2+z^2)=0$

(d) $f(x^3+y^3+z^3, xyz)=0$

24. The solution of $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2x^2}$ is

(a) $f\left(\frac{y}{2}, x+y+z\right)=0$

(b) $f\left(\frac{x}{2}, x^2+y^2+z^2\right)=0$

(c) $f\left(\frac{x}{2}, xyz\right)=0$

(d) $f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right)=0$

25. The solution of $\frac{dx}{x^2-y^2}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$

(a) $\phi\left(x^2+y^2+z^2, \frac{y-z}{z-x}\right)=0$

(b) $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right)=0$

(c) $\phi\left(\frac{x-y}{y-z}, x+y+z\right)=0$

(d) $\phi\left(\frac{x-y}{y-z}, xyz\right)=0$

26. The solution of $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ is

(a) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1,\ xyz=c_2$

(b) $xy+yz+zx=c_1,\ xyz=c_2$

(c) $x^2+y^2+z^2=c_1,\ xyz=c_2$

(d) $(x+y+z)=c_1,\ xyz=c_2$

27. If $t\frac{dx}{dt}+y=0,\ t\frac{dy}{dt}+x=0$ then x is

(a) c_1t+c_2/t (b) c_1t+c_2t

(c) $c_1t^2+c_2t$ (d) $(c_1+c_2t)t$

28. The solution of $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ is

(a) $f(x-y+z, y+z)=0$

(b) $f(x+y+z, xyz)=0$

(c) $f(x+y+z, xy)=0$

(d) $f(x^2+y^2+z^2, x+y+z)=0$

29. The solution of $\frac{dx}{y}=\frac{dy}{x}=\frac{dz}{z}$ is

(a) $x+y=c_1z,\ x^2+y^2=c_2$

(b) $x+z=c_1y,\ x^2-y^2=c_2$

(c) $x+y=c_1z,\ x^2-y^2=c_2$

(d) $x+z=c_1y,\ x^2+y^2=c_2$

30. The solution of $\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$ is

(a) $lx+my+nz=c_1,\ xyz=c_2$

(b) $lx+my+nz=c_1,\ x+y+z=c_2$

(c) $lx+my+nz=c_1,\ e^{xy}=c_2z$

(d) $lx+my+nz=c_1, x^2+y^2+z^2=c_2$

31. The solution of $\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}$ is

(a) $f(xy-z^2, x^2+y^2+z^2)=0$

(b) $f(xy-z^2, x^2-y^2-z^2)=0$

(c) $f(xyz, x^2-y^2-z^2)=0$

(d) $f(xy-z^2, xyz)=0$

32. The solution of $\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$ is

(a) $f\left(zx-zy, \frac{x+y+z}{z}\right)=0$

(b) $f\left(zx-zy, \frac{x+y}{z}\right)=0$

(c) $f\left(xy-yz, \frac{x+y+z}{z}\right)=0$

(d) $f(xy-yz, x-y+z)=0$

33. If $\frac{dy}{dt}=x$ and $\frac{dx}{dt}=y+e^{2t}$ then the value of x is

(a) $c_1e^t+c_2e^{-t}+\frac{2}{3}e^{2t}$ (b) $c_1e^t+c_2e^{-t}+e^{2t}$

(c) $c_1e^t+c_2e^{-t}+\frac{1}{3}e^{2t}$ (d) None of these

34. If $\frac{dx}{dt}+2x=3y$ and $\frac{dy}{dt}+2y=3x$ then x is

(a) $c_1e^t+c_2e^{3t}$ (b) $c_1e^t+c_2e^{-3t}$

(c) $c_1e^t+c_2e^{5t}$ (d) $c_1e^t+c_2e^{-5t}$

35. If $\frac{dx}{dt}+\frac{2dy}{dt}-2x+2y=0$ and

$3\frac{dx}{dt}+\frac{dy}{dt}+2x+y=0$ then x is

(a) $ce^{-6/5t}$ (b) $c_1e^t+c_2e^{-2t}$

(c) $c_1e^{2t}+c_2e^{-3t}$ (d) $ce^{6/5t}$

36. If $\frac{dy}{dt}=x$ and $\frac{dx}{dt}=y$ then x is

(a) $c_1\cos t+c_2\sin t$ (b) $(c_1+c_2t)e^t$

(c) $c_1e^t+c_2e^{-t}$ (d) $(c_1+c_2t)e^{-t}$

37. If $\frac{dx}{dt}+\frac{2}{t}(x-y)=0$ and $\frac{dy}{dt}+\frac{1}{t}(x+5y)=0$ then differential equation for x is
(a) $(t^2D^2-4tD+6)x=0$
(b) $(t^2D^2-8D+8)x=0$
(c) $(t^2D^2-4tD+12)x=0$
(d) $(t^2D^2+8tD+12)x=0$

38. If we choose 1, x, z or multipliers in $\frac{xdx}{z^2-2yz-y^2}=\frac{dy}{y+z}=\frac{dz}{y-z}$ then
(a) $x+y+z=c$
(b) $x^2+y^2+z^2=c$
(c) $xyz=c$
(d) None of these

39. If we choose x, $-y, -z$ as multipliers in $\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}$ then
(a) $x^2+y^2+z^2=c$
(b) $x^2-y^2+z^2=c$
(c) $x^2+y^2-z^2=c$
(d) $x^2-y^2-z^2=c$

40. By choosing x, y, z as multipliers in $\frac{dx}{x(y^2-z^2)}=\frac{dy}{-y(z^2+x^2)}=\frac{dz}{z(x^2+y^2)}$, we have
(a) $x^2+y^2+z^2=c$
(b) $x^2y^2z^2=c$
(c) $xyz=c$
(d) $x+y+z=c$

41. By choosing $\frac{1}{x},\frac{1}{y},\frac{1}{3z}$ as multipliers in $\frac{dx}{y^3x-2x^4}=\frac{dy}{2y^4-x^3y}=\frac{dz}{9z(x^3-y^3)}$ we have
(a) $xyz=c$
(b) $x^{1/3}yz=c$
(c) $xy^{1/3}z=c$
(d) $xyz^{1/3}=c$

42. Elimination of y between $t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+2y=0$, $t^2\frac{d^2y}{dx^2}+t\frac{dy}{dt}-2x=0$ gives
(a) $(D^4+4)x=0$
(b) $(D^4+2D^2+3)x=0$
(c) $(D^3+4)x=0$
(d) $(D^2+4)x=0$

43. The differential equation in x from $3\frac{dx}{dt}+\frac{2dy}{dt}-4x+3y=0$ and $4\frac{dx}{dt}+\frac{dy}{dt}+3x+4y=0$ is
(a) $(D^2+D+3)x=0$
(b) $(D^2+3D+5)x=0$
(c) $(D^2-2D+5)x=0$
(d) $(D^2+2D+5)x=0$

44. The solution of $\frac{dx}{dt}+\frac{dy}{dt}+2x+y=0$ and $\frac{dy}{dt}+5x+3y=0$ for x is
(a) $(c_1+c_2t)e^t$
(b) $c_1\cos t+c_2\sin t$
(c) $c_1e^t+c_2e^{-t}$
(d) $(c_1+c_2t)\cos t$

45. The differential equation in x from $\frac{d^2x}{dt^2}-\frac{dy}{dt}=2x+2t$ and $\frac{dx}{dt}+4\frac{dy}{dt}=3y$ is
(a) $(D^3+D^2-D+3)x=4-3t$
(b) $(D^3-D^2-4D+1)x=4-3t$
(c) $(D^2-3D+4)x=4-3t$
(d) $(2D^3-D^2-4D+3)x=4-3t$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(a)	**3.**	(d)	**4.**	(d)	**5.**	(a)	**6.**	(a)	**7.**	(d)	**8.**	(c)	**9.**	(a)	**10.**	(d)
11.	(c)	**12.**	(b)	**13.**	(d)	**14.**	(d)	**15.**	(a)	**16.**	(d)	**17.**	(c)	**18.**	(d)	**19.**	(a)	**20.**	(b)
21.	(c)	**22.**	(d)	**23.**	(a)	**24.**	(d)	**25.**	(b)	**26.**	(c)	**27.**	(a)	**28.**	(b)	**29.**	(c)	**30.**	(d)
31.	(b)	**32.**	(a)	**33.**	(a)	**34.**	(d)	**35.**	(a)	**36.**	(c)	**37.**	(d)	**38.**	(b)	**39.**	(d)	**40.**	(a)
41.	(d)	**42.**	(a)	**43.**	(d)	**44.**	(b)	**45.**	(d)										

HINTS AND SOLUTIONS

1. Given $\frac{dx}{dt} = -\omega y$, differentiate it w.r.t. t

$$\frac{d^2x}{dt^2} = -\omega\frac{dy}{dt} = -\omega(\omega x) = -\omega^2 x$$

So $(D^2+\omega^2)x = 0$ A.E. is $m^2+\omega^2 = 0$

or $m = \pm\omega i$ so $x = A\cos\omega t + B\sin\omega t$

3. Given $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} = t^2 - 5$

Differentiate first equation w.r.t. t we get

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 1 \Rightarrow \frac{d^2x}{df^2} - t^2 + 5 = 1$$

i.e. $\frac{d^2x}{dt^2} = t^2 - 4$ integrating it

$$\frac{dx}{dt} = \frac{t^3}{3} - 4t + A \text{ again integrating}$$

$$x = \frac{t^4}{12} - 2t^2 + At + B$$

5. $Dx = 2y, Dy = 2z, Dz = 2x$

Differentiate first equation $D^2x = 2Dy = 4z$

Again differentiate $D^3x = 4Dz = 8x \Rightarrow (D^3 - 8)x = 0$

6. Given that $D^2y = x$ and $D^2x = y$

Differentiate first equation twice we get

$D^4y = D^2x = y \Rightarrow (D^4 - 1)y = 0$

$\Rightarrow \quad (D^2+1)(D^2-1)y = 0$

So A. E. is $(m^2+1)(m^2-1) = 0 \Rightarrow m = 1, -1, i, -i$

So $y = Ae^t + Be^{-t} + C\cos t + D\sin t$

9. Given that $D^2x + y = \sin t$ ...(1)

and $x + D^2y = \cos t$ ...(2)

Multiplying equation (1) by D^2 and then subtract equation (2)

$$(D^2-1)x = -\sin t - \cos t$$

11. Given that

$$(D+2)x + (D+1)y = 0 \quad \text{...(1)}$$
$$5x + (D+3)y = 0 \quad \text{...(2)}$$

Multiplying (1) by $(D+3)$, (2) by $(D+1)$ and then subtracting, we get

$$[(D+2)(D+3) - 5(D+1)]x = 0$$

or $(D^2+1)x = 0$, A.E. is $m^2+1 = 0, m = \pm i$

So solution is $x = A\cos t + B\sin t$

13. Given $x\frac{dy}{dx} = -z$ and $x\frac{dz}{dx} = -y$

Differentiate equation first we get

$$+x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{dz}{dx} = \frac{y}{x}$$

i.e. $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

14. Given $t\,dx = (t-2x)\,dt$

$$\frac{dx}{dt} = 1 - \frac{2x}{t} \Rightarrow \frac{dx}{dt} + \frac{2}{t}x = 1$$

I.F. $e^{\int 2/t\,dt} = t^2$, So solution is $xt^2 = \int t^2 dt + A$

$$xt^2 \pm \frac{t^3}{3} + A \quad \Rightarrow \quad x = \frac{t}{3} + \frac{A}{t^2}$$

17. Given $(D^2-3)x - 4y = 0$ ...(1)

and $x + (D^2+1)y = 0$ ...(2)

Multiplying (2) by (D^2-3) and then subtracting, we get

or $(D^2-1)^2 y = 0$

18. After solving $x = A\cos\omega t + B\sin\omega t$ and $y = A\sin\omega t - B\cos\omega t$

$x^2 + y^2 = A^2 + B^2$ i.e. a circle

20. Given that $\frac{dx}{y^2z/x} = \frac{dy}{xz} = \frac{dz}{yz}$

By first and second term $x^2dx - y^2dy = 0$ $\Rightarrow x^3 - y^3 = c_1$

and by second and third term $ydy - zdz = 0$ $\Rightarrow y^2 - z^2 = c_2$

21. Given $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$

Consider $\frac{xdx + ydy + zdz}{0}$ we get $x^2 + y^2 + z^2 = c_1$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier we have each fraction

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0} \text{ so } \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\log xyz = \log c_2 \Rightarrow xyz = c_2$

22. Given that $Dx + y = \sin t$ ...(1)

$x + Dy = \cos t$ ...(2)

Multiplying (1) by D and subtracting (2) we get

$$(D^2 - 1)x = \cos t - \cos t = 0$$

A.E. is $m^2 - 1 = 0 \Rightarrow m = 1, -1$

So solution is $x = c_1e^t + c_2e^{-t}, x' = c_1e^t - c_2e^{-t}$

$\because$ $x(0) = 0, x'(0) = 1$ so we get $0 = c_1 + c_2$ and

$1 = c_1 - c_2 \Rightarrow c_1 = 1, c_2 = -1$ so we get

$x = e^t - e^{-t}$

24. $$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking last two terms we get $\frac{dy}{y} = \frac{dz}{z}$

$\Rightarrow \log y = \log z + \log c_1$

$$\frac{y}{z} = c_1$$

consider $\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$

Integrating it $\log(x^2 + y^2 + z^2) = \log z + \log c_2$

i.e. $$\frac{x^2 + y^2 + z^2}{z} = c_2$$

So solution is $f\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$

27. Given that $t\frac{dx}{dt} + y = 0$ and $t\frac{dy}{dt} \neq x = 0$

Differentiating first equation w.r.t. t we get

$$\frac{td^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0 \Rightarrow t\frac{d^2x}{dt^2} + \frac{dx}{dt} - \frac{x}{t} = 0$$

i.e. $$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$$

put it e^z we get $[D(D-1) + D - 1]x = 0\, D \equiv \frac{d}{d_2}$

$(D^2 - 1)x = 0$ A.E. $m^2 - 1 = 0 \Rightarrow m = 1, -1$

So solution $x = c_1e^z + c_2e^{-z} = c_1t + c_2/t$

35. Given that $(D - 2)x + (2D + 2)y = 0$ ...(1)

and $(3D + 2)x + (D + 1)y = 0$ ...(2)

Multiplying (2) by 2 and subtracting from (1) we get

$(5D + 6)x = 0$ A.E. is $5m + 6 = 0 \Rightarrow m = -\frac{6}{5}$

So solution is $x = ce^{-6/5t}$

37. Given $t\frac{dx}{dt} + 2x - 2y = 0$ and $\frac{dy}{dt} = -\frac{x + 5y}{t}$

Differentiating first equation w.r.t. t we get

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{2dx}{dt} - \frac{2dy}{dt} = 0$$

or $$t\frac{d^2x}{dt^2} + \frac{3dx}{dt} + \frac{2(x + 5y)}{t} = 0$$

or $$t^2\frac{d^2x}{dt^2} + \frac{3t\,dx}{dt} + 2\left(\frac{2 + 5/2 + dx/dt + 5x}{t}\right) = 0$$

After solving we get $(t^2D^2 + 8tD + 12)x = 0$

39. Given $\frac{dx}{z(x + y)} = \frac{dy}{z(x - y)} = \frac{dz}{x^2 + y^2}$

Choosing $x, -y, -z$ as multipliers, we get

$$\frac{xdx - ydy - zdz}{x^2z + xyz - xyz + y^2z - x^2z - y^2z}\ i.e.$$

$$xdx - ydy - zdz = 0$$

integrating $x^2 - y^2 - z^2 = c$

41. Given that $\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{3z}$ as multipliers we get

$$\frac{\frac{dy}{x} + \frac{dy}{y} + \frac{dz}{3z}}{y^3 - 2x^3 + 2y^3 - x^3 + 3x^3 - 3y^3}$$

i.e. $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z} = 0$, integrating $xyz^{1/3} = c$

43. Given $(3D - 4)x + (2D + 3)y = 0$ ...(1)

and $(4D + 3)x + (D + 4)y = 0$ ...(2)

Multiplying (1) by $(D + 4)$ and (2) by $(2D + 3)$ and subtracting, we get

$$[(3D - 4)(D + 4) - (4D + 3)(2D + 3)]x = 0$$

i.e. $$(D^2 + 2D + 5)y = 0$$

45. Given equations are

$(D^2 - 2)x - Dy = 2t$ ...(1)

and $Dx + (4D - 3)y = 0$ ...(2)

Eliminating y from (1) and (2), we get

$$[(4D - 3)(D^2 - 2) + D^2]x = (4D - 3)2t$$

i.e. $$(2D^3 - D^2 - 4D + 3)x = 4 - 3t$$

* * *

Unit-II

CHAPTER

7

Linear Equations of Second Order with Variable Coefficients

LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

1. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q and R are functions of x only, is called the linear differential equation of second order.
2. If an integral included in the complementary function of such an equation be known then the general solution can be found in terms of the known integral.
3. Let $y = x$ be known integral *i.e.* solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

and $y = uv$ be the solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

then after solving, we get

$$y = c_1u\int\frac{e^{-\int Pdx}}{u^2}\,dx + c_2u$$

$$+u\int\left[\frac{e^{-\int Pdx}}{u^2}\int u\,Re^{\int Pdx}dx\right]dx$$

Where last term shows the particular integral.

4. The solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ can be find by the following results.

(i) $y = e^x$ is a part of complementary function if $1 + P + Q = 0$

(ii) $y = e^{-x}$ is a part of (C.F.) if $1 - P + Q = 0$

(iii) $y = e^{mx}$ is part of C.F. if $m^2 + mP + Q = 0$

(iv) $y = x$ is a part of C.F. if $P + Qx = 0$

(v) $y = x^2$ is a part of C.F. if $2 + 2Px + Qx^2 = 0$

(vi) $y = x^m$ is a part of C.F. if $m(m-1) + Pmx + Qx^2 = 0$

REMOVAL OF THE FIRST DERIVATIVE (REDUCTION TO NORMAL FORM)

1. If we are unable to obtain a part of C.F. of the solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

then we change the dependent variable from y to v in equation (1) by putting $y = uv$ (where u is a function of x) we get

$$u\frac{d^2v}{dx^2} + u\left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}$$

$$+v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = R \qquad ...(2)$$

To remove the first derivative term, we take

$$P + \frac{2}{u}\frac{du}{dx} = 0$$

$$\Rightarrow \qquad u = e^{-\frac{1}{2}\int Pdx} \qquad ...(3)$$

So equation (2) becomes

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}\right)v = Re^{\frac{1}{2}\int Pdx}$$

$$i.e. \qquad \frac{d^2v}{dx^2} + Xv = Y \qquad ...(4)$$

where $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and $Y = Re^{\frac{1}{2}\int Pdx}$

Equation (4) is called the normal form of equation (1) and can be solve easily.

TRANSFORMATION OF THE EQUATION BY CHANGING THE INDEPENDENT VARIABLE x

1. Let the differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Then by changing the independent variable be changed from x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\text{where } P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\; Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2},$$

$$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

where P_1, Q_1 and R_1 are functions of x.

Choose $\left(\dfrac{dz}{dx}\right)^2 = Q$ or variable part of Q such that it is a perfect square then $\dfrac{dz}{dx} = \sqrt{Q}$ then calculate P_1, Q_1 and R_1.

If P_1 is zero or constant, then this method is applicable.

METHOD OF VARIATION OF PARAMETERS

Let the linear equation of second order be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

and let $y = Au + Bv$

Is the complementary function of (1), where u and v are functions of x and A and B are constants.

So $(Au_2 + Bv_2) + P\,(Au_1 + Bv_1)$

$$+Q\,(Au + Bv) = 0$$

i.e. $$u_2 + Pu_1 + Qu = 0$$

and $$v_2 + Pv_1 + Qv = 0 \qquad ...(2)$$

Again let $y = Au + Bv$ ...(3)

is the complete primitive of (1) then A and B are functions of x so by (3)

$$\frac{dy}{dx} = A_1 + B_1 v + Au_1 + Bv_1$$

Choose A and B such that

$$A_1 u + B_1 v = 0 \qquad ...(4)$$

So by (1), (2) and (4) we get

$$A_1 u_1 + B_1 v_1 = R_1 \qquad ...(5)$$

Solving (4) and (5), we get

$$A_1 = -\frac{vR}{u_1 v - uv_1} \text{ and } B_1 = \frac{uR}{v_1 u - vu_1}$$

so $$A = \int\frac{vR\,dx}{u_1 v - uv_1} + C_1$$

and $$B = \int\frac{uR}{v_1 u - vu_1}dx + C_2$$

Substitute A and B in (3) we get the complete primitive of (1).

Now, consider $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Then P.I. of (1) is

$$-u\int\frac{vR}{w}dx + v\int\frac{uR}{w}dx$$

$$= -u\int\frac{vR}{uv_1 - vu_1}dx + v\int\frac{uR}{uv_1 - vu_1}dx \qquad ...(2)$$

Where u and v are defined by $y = c_1 u + c_2 v$

and Wronskian of u and v is $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix}$

∴ C.S. = C.F. + P.I.

METHOD OF OPERATIONAL FACTORS

Let the linear equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

i.e. $f(D)\,y = R \;\Rightarrow\; f_1(D)f_2(D)y = R$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. $y = e^x$ is the part of solution of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$ if :

(a) $1 - P + Q = 0$ (b) $P + Qx = 0$

(c) $1 + P + Q = 0$ (d) $Px + Q = 0$

2. For $P\dfrac{d^2y}{dx^2} + Q\dfrac{dy}{dx} + Ry = X$, if $P - Q + R = 0$, then a part of the C.F. is :

(a) $y = e^{-x}$ (b) $y = e^x$

(c) $y = x$ (d) $y = x^2$

3. The part of C.F. of the differential equation :

$$x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = 0 \text{ is}$$

(a) $y = e^{-x}$ (b) $y = x$

(c) $y = x^2$ (d) $y = e^x$

4. $y = x$ is a part of complementary function of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ if :

(a) $1 + P + Q = 0$ (b) $1 - P + Q = 0$

(c) $Q + Px = 0$ (d) $P + Qx = 0$

5. A part of the complementary function of the differential equation $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0$ is :

(a) x (b) x^2 (c) e^x (d) e^{2x}

6. The part of C.F. of the differential equation

$$x\dfrac{dy}{dx} - y = (x-1)\left(\dfrac{d^2y}{dx^2} - x + 1\right) \text{ is :}$$

(a) $y = x^2$ (b) $y = e^{-x}$

(c) $y = e^x$ (d) $y = x$

7. A part of the C.F. of the solution of the differential equation $\dfrac{d^2y}{dx^2} + (1 - \cot x)\dfrac{dy}{dx} - y\cot x = \sin^2 x$ is :

(a) $y = e^{-x}$ (b) $y = e^x$

(c) $y = x$ (d) $y = x^3$

8. A part of the C.F. of the solution of the differential equation $\dfrac{d^2y}{dx^2} - \dfrac{2(x+1)}{x}\dfrac{dy}{dx} + \dfrac{x+2}{x}y = \dfrac{x-2}{x}e^x$ is :

(a) $y = x$ (b) $y = e^x$ (c) $y = x^2$ (d) $y = e^{-x}$

9. If $y = e^{-x}$ is a part of C.F. of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} - \dfrac{2}{x}y = x^2$ then value of P is :

(a) $x - 2$ (b) $x + 2$

(c) $\dfrac{x-2}{x}$ (d) $\dfrac{x+2}{x}$

10. If $y = e^x$ is a part of C.F. of $x\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + Qy = e^x$ then Q is :

(a) $2x - 1$ (b) $2x + 1$

(c) $+1$ (d) -1

11. A part of C.F. of $x^2\dfrac{d^2y}{dx^2} - 2x\,(1+x)\dfrac{dy}{dx} + 2\,(1+x)\,y = x^3$ is :

(a) $y = x$ (b) $y = x^3$

(c) $y = e^x$ (d) $y = e^{2x}$

12. $y = e^{-x}$ is a part of C.F. of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ if :

(a) $1 - P + Q = 0$ (b) $1 + P + Q = 0$

(c) $P + Q = 0$ (d) $P - Q = 0$

13. The solution of $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$, given that $y = x^3$ is a solution is :

(a) $y = c_1x^3 + \dfrac{c_2}{x}$ (b) $y = c_1x^3 + \dfrac{c_2}{x^2}$

(c) $y = c_1x^3 + \dfrac{c_2}{x^3}$ (d) None of these

14. A part of C.F. of $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = x$ is :

(a) $y = e^x$ (b) $y = e^{-x}$

(c) $y = x$ (d) $y = x^2$

15. In reducing the equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ in the normal form $\dfrac{d^2v}{dx^2} + Xv = Y$, value of X is :

(a) $e^{-\frac{1}{2}\int Pdx}$ (b) $Re^{\frac{1}{2}\int Pdx}$

(c) $Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{P^2}{4}$ (d) None of these

16. The normal form of the differential equation $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)\,y = e^{x^2}$ is :

(a) $(D^2 + 1)\,v = 1$ (b) $(D^2 + 1)\,v = x$

(c) $(D^2 - 1)\,v = e^x$ (d) $(D^2 + 2)\,v = 1$

17. To change the equation $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 + 1)\,y = x^3 + 3x$ in normal form the value of u is :

(a) e^x (b) e^{x^2} (c) $e^{-x^2/2}$ (d) e^{-x^2}

18. In reducing the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ in the normal form the value of u is :

(a) $e^{\frac{1}{2}\int Pdx}$ (b) $e^{-\frac{1}{2}\int Pdx}$

(c) $e^{-\frac{1}{2}\int Rdx}$ (d) $e^{-\frac{1}{2}\int Qdx}$

19. $y = e^{mx}$ is a part of C.F. of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ if :

(a) $2 + 2Px + Qx^2 = 0$ (b) $m^2 + Pm + Q = 0$

(c) $mP + Qx = 0$ (d) $P + mQ + m^2 = 0$

20. The part of C.F. of the differential equation $(2x - 1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + (3 - 2x)\,y = 2e^x$ is

(a) $y = x$ (b) $y = x^2$ (c) $y = e^{-x}$ (d) $y = e^x$

21. If $y = \cot x$ is a solution of $\sin^2 x\frac{d^2y}{dx^2} = 2y$ then complete solution is :

(a) $y = c_1 \cot x + c_2 x \cot x$

(b) $y = c_1 \cot x + (1 - x\cot x)\,c_2$

(c) $y = c_1 \cot x + c_2 \sin x$

(d) $y = c_1 \cot x + (1 + \cot x)\,c_2$

22. In reducing the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ in the normal form $\frac{d^2v}{dx^2} + Xv = Y$, value of Y is :

(a) $Re^{\frac{1}{2}\int Pdx}$ (b) $Re^{-\frac{1}{2}\int Pdx}$

(c) $e^{\frac{1}{2}\int Pdx}$ (d) $e^{-\frac{1}{2}\int Pdx}$

23. To change the differential equation $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\,y = -3e^{x^2}\sin 2x$ in normal form the value of u is :

(a) e^x (b) e^{x^2} (c) e^{-x} (d) e^{-x^2}

24. In reducing the equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$$

in normal form $\frac{d^2v}{dx^2} + Xv = Y$ then X is :

(a) x (b) 1 (c) e^x (d) $\frac{1}{x}$

25. In reducing the equation $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + y\cos^2 x = 0$ in $\frac{d^2v}{dx^2} + Xv = y$, value of y is :

(a) $\sec x$ (b) 2 (c) $2x$ (d) 0

26. If the equation $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 0$ is changed to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = 0$ then P_1 is :

(a) 0 (b) -2 (c) $2\cos^2 x$ (d) x

27. If the equation $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 0$ is transformed into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = 0$ then Q_1 is :

(a) -1 (b) $2\sin z$

(c) $-x$ (d) None of these

28. If the equation $\frac{d^2y}{dx^2} + (\tan x - 1)^2\frac{dy}{dx} - n(n-1)y\sec^4 x = 0$ is transformed to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q, y = 0$ then P_1 is :

(a) 1 (b) $n\,(n - 1)$

(c) $-n\,(n - 1)$ (d) None of these

29. The transformed equation of $x^6\frac{d^2y}{dx^2} + 3x^5\frac{dy}{dx} + a^2y = \frac{1}{x^2}$ from x to z variable is :

(a) $(D^2 - a^2)\,y = 0$ (b) $(D^2 + a^2)\,y = z$

(c) $(D^2 + a^2)\,y = -2z$ (d) $(D^2 - a^2)\,y = -2z$

30. If $y = A\cos 2x + B\sin 2x$ be the complete primitive of $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ then Wronskian is :

(a) 1 (b) 2 (c) 3 (d) 4

31. The Wronskian of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$ is :

(a) e^x (b) e^{2x} (c) 1 (d) 0

32. The Wronskian of $\frac{d^2y}{dx^2} + n^2y = \sec nx$ is :

(a) x^2 (b) $\sec nx$ (c) nx (d) n

33. The Wronskian of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$ is :

(a) 1 (b) $-\frac{2}{x}$ (c) $\frac{2}{x}$ (d) $\frac{1}{x}$

34. If $y = vx$ be the solution of $x^2\frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x + 2)\ y = x^3e^x$ then the transformed equation is :

(a) $(D^2 - 1)\ v = e^x$ (b) $(D^2 + 1)\ v = e^x$

(c) $(D^2 - 1)\ v = e^{-x}$ (d) $(D^2 + 1)\ v = e^{-x}$

35. If $y = vx^3$ be the solution of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$ then transformed equation is :

(a) $(D^2 + 7D)\ v = 0$ (b) $\left(D^2 + \frac{7}{x}D\right)v = 0$

(c) $(D^2 - 7D)\ v = 0$ (d) $\left(D^2 - \frac{7}{x}D\right)v = 0$

36. If $y = ve^x$ be the solution of $x\frac{d^2y}{dx^2} - (2x - 1) + (x - 1) = 0\frac{dy}{dx}$ then transformed equation is :

(a) $(xD^2 - D)\ v = 0$ (b) $(xD^2 + D)\ v = 0$

(c) $(xD^2 + 1)\ v = 0$ (d) $(xD^2 - 1)\ v = 0$

37. If $y = v\sec x$ be the solution of $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = \sec xe^x$ then transformed equation is :

(a) $(D^2 + 6)\ v = e^x$ (b) $(D^2 - 6)\ v = e^x$

(c) $(D^2 + 5)\ v = xe^x$ (d) $(D^2 + 6)\ v = \sec xe^x$

38. To change the equation $x^2\frac{d^2y}{dx^2} - 2\,(x^2 + x)\frac{dy}{dx} + (x^2 + 2x + 2)\ y = 0$ into normal form, the value of u is :

(a) x (b) e^x (c) xe^{-x} (d) xe^x

39. The part of C.F. of the differential equation $(3 - x)\frac{d^2y}{dx^2} - (9 - 4x)\frac{dy}{dx} + (6 - 3x)\ y = 0$ is :

(a) $y = x$ (b) $y = x^2$ (c) $y = e^x$ (d) $y = e^{2x}$

40. If the equation $\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$ be transformed into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ then R_1 is :

(a) 1 (b) e^{-z} (c) e^{+z} (d) -1

41. If the equation $\frac{d^2y}{dx^2} - (1 + 4e^x)\frac{dy}{dx} + 3e^{2x} = e^{2(x+e^x)}$ be transformed into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ then P_1 is :

(a) 3 (b) 4 (c) e^{2z} (d) e^{2x}

42. The differential equation $[3x^2D^2 + (2 + 6x - 6x^2)D - 4]\ y = 0$ is equivalent to :

(a) $(3x^2D + 2)\ (D - 2)\ y = 0$

(b) $(3x^2D + 2)\ (D + 2)\ y = 0$

(c) $(D - 2)\ (3x^2D + 2)\ y = 0$

(d) $(D + 2)\ (3x^2D - 2)\ y = 0$

43. The differential equation $[xD^2 - (x + 2)\ D + 2]\ y = x^3$ is equivalent to :

(a) $(D - 1)\ (xD - 2)\ y = x^3$

(b) $(D + 1)\ (xD + 2)\ y = x^3$

(c) $(xD - 2)\ (D + 1)\ y = x^3$

(d) $(xD - 2)\ (D - 1)\ y = x^3$

44. The differential equation $x\frac{d^2y}{dx^2} + (x - 1)\frac{dy}{dx} - y = x^2$ is equivalent to :

(a) $(xD + 1)\ (D - 1)\ y = x^2$

(b) $(D - 1)\ (xD - 1)\ y = x^2$

(c) $(xD - 1)\ (D + 1)\ y = x^2$

(d) $(D + 1)\ (xD + 1)\ y = x^2$

45. The differential equation $[3x^2D^2 + (2 - 6x^2)D - 4]y = 0$ is equivalent to :

(a) $(3x^2D + 2)\ (D + 2)\ y = 0$

(b) $(3x^2D + 2)\ (D - 2)\ y = 0$

(c) $(D + 2)\ (3x^2D + 2)\ y = 0$

(d) None of these

46. The differential equation $[xD^2 + (x - 2)\ D - 2]\ y = 0$ is equivalent to :

(a) $(xD - 2)\ (D + 1)\ y = 0$ (b) $(xD + 2)\ (D + 1)\ y = 0$

(c) $(D+1)(xD-2)\,y=0$ (d) $(D-1)(xD+2)\,y=0$

47. If $y = A\cos x + B\sin x$ be the general solution of $(D^2+1)\,y = x$ then A is :

(a) $x\sin x - \cos x + c$ (b) $x\cos x - \sin x + c$

(c) $x\cos x + \sin x + c$ (d) $x\sin x + \cos x + c$

48. If $y = A\cos ax + B\sin ax$ be the general solution of $(D^2+a^2)\,y = \operatorname{cosec} ax$ then A is :

(a) $a_x + c$ (b) $\frac{x}{a} + c$ (c) $-\frac{x}{a} + c$ (d) $-ax + c$

49. If $y = Au + Bv$ be the complete primitive of $D^2y + PDy + Qy = R$ then A and B are functions of :

(a) x and y (b) x only (c) y only (d) Constant

50. If $y = Ax + Bxe^{2x}$ be the complete primitive of $[x^2D^2 - 2x(1+x)D + 2(1+x)]\,y = x^3$ then A is :

(a) $x + c$ (b) $\frac{x}{2} + c$ (c) $-\frac{x}{2} + c$ (d) $-x + c$

51. The normal form of $(D^2 + PD + Q)\,y = R$ is :

(a) $(D + x)v = Y$ (b) $(D^2 + xD)v = 0$

(c) $(D + xD)v = 0$ (d) $(D^2 + x)v = Y$

52. The normal form of $\left[D^2 - \frac{2}{x}D + \left(1 + \frac{2}{x^2}\right)\right]y = 0$ is

(a) $(D^2+1)\,v = x$ (b) $(D^2+1)\,v = 0$

(c) $(D^2-1)\,v = x$ (d) $(D^2-1)\,v = 0$

53. The equation $[x^2D^2 + D - (1+x^2)]\,y = 0$ is equivalent to :

(a) $(D-1)\,[x^2(D+1)+1]\,y = 0$

(b) $[x^2(D-1)-1]\,(D+1)\,y = 0$

(c) $[x^2(D-1)+1]\,(D+1)\,y = 0$

(d) $[x^2(D+1)+1]\,(D-1)\,y = 0$

54. If $y = A\cos x + B\sin x$ be the general solution of $(D^2+1)\,y = x$ then B is :

(a) $x\sin x + \cos x + c$ (b) $x\cos x + \sin x + c$

(c) $x\sin x - \cos x + c$ (d) $x\cos x - \sin x + c$

55. If $y = A\cos 2x + B\sin 2x$ be the general solution of $(D^2+4)\,y = 4\tan 2x$ then B is :

(a) $-\sin 2x + c$ (b) $-\tan 2x + c$

(c) $-\cos 2x + c$ (d) $\cos 2x + c$

56. If $\left(D^2 + \frac{2}{x}D + n^2\right)y = 0$ is reduced to normal form $(D^2 + X)\,v = Y$ then X is :

(a) 0 (b) n (c) n^2 (d) $\frac{1}{x}$

57. If $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is transformed to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ then P_1 is :

(a) $\dfrac{\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$ (b) $\dfrac{\left(\frac{d^2z}{dx^2} + \frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$

(c) $\dfrac{\frac{d^2z}{dx^2} - P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ (d) $\dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$

58. If $y = e^x$ is a part of C.F. of $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - Qy = e^x\sin x$ then Q is :

(a) $1 + \cot x$ (b) $\cot x$

(c) $1 - \cot x$ (d) $e^x\cot x$

59. If $y = A\cos ax + B\sin ax$ be the general solution of $(D^2+a^2)\,y = \sec ax$ then B is :

(a) $ax + c$ (b) $x + c$

(c) $\frac{x}{a} + c$ (d) $\frac{a}{x} + c$

60. If $y = Ax + \frac{B}{x}$ be the general solution of $(x^2D^2 + xD - 1)\,y = x^2e^x$ then A is :

(a) $\frac{e^{-x}}{2}$ (b) e^x (c) e^{-x} (d) $\frac{e^x}{2}$

61. If $y = ve^x$ be the solution of $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = e^x$ then transformed equation is :

(a) $x\frac{d^2v}{dx^2} + (1+x)\frac{dv}{dx} = 1$

(b) $\frac{d^2v}{dx^2} + \left(\frac{1+x}{x}\right)\frac{dv}{dx} = 0$

(c) $x\frac{d^2v}{dx^2} + (1-x)\frac{dv}{dx} = 1$

(d) $\frac{d^2v}{dx^2} + \left(\frac{1-x}{x}\right)\frac{dv}{dx} = 0$

62. If $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0$ is transformed into normal form $\frac{d^2v}{dx^2} + Xv = Y$ then X is :

(a) 0 (b) a (c) a^2 (d) $\frac{2}{a^2}$

63. If $\frac{d^2y}{dx^2} + 2\tan x \frac{dy}{dx} + (1 + 2\tan^2 x)\, y = \sec x \tan x$ is transformed into normal form $\frac{d^2v}{dx^2} + Xv = Y$ then y is :
(a) $\tan x$ (b) $\sec^2 x$
(c) $\sec x \tan x$ (d) $\sec^2 x \tan x$

64. In reducing $(D^2 - 2\tan x + 5)\, y = 0$ into normal form choosen such that :
(a) $u = \tan x$ (b) $u = \sec x$
(c) $u = \cot x$ (d) $u = \operatorname{cosec} x$

65. If the equation $\left(D^2 - \frac{1}{x}D + 4x^2\right) y = x^4$ be transformed into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ then P_1 is :
(a) 0 (b) 2 (c) 4 (d) 1

66. If the equation $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y\sin^2 x = 0$ be transformed into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ then P_1 is :
(a) 0 (b) 1 (c) –1 (d) 2

67. If $x^2\frac{dy}{dx} + 2xy = \frac{2\ln x}{x}$ and $y\,(1) = 0$, then $y(e)$ is :
(a) 1 (b) $\frac{1}{e}$ (c) e (d) $\frac{1}{e^2}$

68. The solution of $x^2\frac{dy}{dx} + 2xy - x + 1 = 0$ when $x = 1, y = 0$ is :
(a) $\frac{1}{z} - \frac{1}{x} + \frac{1}{2x^2}$ (b) $\frac{1}{z} + \frac{1}{x} - \frac{1}{2x^2}$
(c) $1 - \frac{1}{x} + \frac{1}{x^2}$ (d) $1 + \frac{1}{x} - \frac{1}{x^2}$

69. If complete solution of $(D^2 + PD + Q)\, y = 0$ is $y = c_1e^{-x} + c_2e^{-3x}$ then P and Q are :
(a) $P = 3, Q = 4$ (b) $P = 4, Q = 3$
(c) $P = 3, Q = 3$ (d) $P = 4, Q = 4$

70. If $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 5$ then $\lim_{t\to 8} x\,(t)$ is :
(a) $\frac{5}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

71. To change the equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + x^2)\, y = 0$ into $\frac{d^2v}{dx^2} + Xv = Y$, X is equal to:
(a) x (b) 1 (c) $\frac{x^2}{2}$ (d) 0

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(a)	**3.**	(d)	**4.**	(d)	**5.**	(c)	**6.**	(d)	**7.**	(a)	**8.**	(b)	**9.**	(c)	**10.**	(d)
11.	(a)	**12.**	(a)	**13.**	(c)	**14.**	(c)	**15.**	(c)	**16.**	(a)	**17.**	(c)	**18.**	(b)	**19.**	(b)	**20.**	(d)
21.	(b)	**22.**	(a)	**23.**	(b)	**24.**	(b)	**25.**	(d)	**26.**	(a)	**27.**	(a)	**28.**	(a)	**29.**	(c)	**30.**	(b)
31.	(b)	**32.**	(d)	**33.**	(b)	**34.**	(a)	**35.**	(b)	**36.**	(b)	**37.**	(a)	**38.**	(d)	**39.**	(c)	**40.**	(c)
41.	(b)	**42.**	(c)	**43.**	(d)	**44.**	(c)	**45.**	(b)	**46.**	(a)	**47.**	(b)	**48.**	(c)	**49.**	(b)	**50.**	(c)
51.	(d)	**52.**	(b)	**53.**	(d)	**54.**	(a)	**55.**	(c)	**56.**	(c)	**57.**	(d)	**58.**	(c)	**59.**	(c)	**60.**	(d)
61.	(a)	**62.**	(c)	**63.**	(d)	**64.**	(b)	**65.**	(a)	**66.**	(a)	**67.**	(d)	**68.**	(a)	**69.**	(b)	**70.**	(a)
71.	(b)																		

HINTS AND SOLUTIONS

3. Given $\frac{d^2y}{dx^2} - \left(\frac{2x+1}{x}\right)\frac{dy}{dx} + \left(\frac{x-1}{x}\right) y = 0$

Comparing $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ we have

$1 + P + Q = 0$ so e^x is a part of C.F.

6. Given that $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}y = x - 1$

after comparing $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ we have

$P + Qx = 0$, so $y = x$ is the part of C.F.

9. If $y = e^{-x}$ is a part of C.F. of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} - \frac{2}{x}y = x^2$

then $1 - P + Q = 0$

or $1 - P - \frac{2}{x} = 0$

⇒ $P = 1 - \frac{2}{x} = \frac{x-2}{x}$

11. Given that $x^2\frac{d^2y}{dx^2} - 2x\,(1+x)\frac{dy}{dx} + 2\,(1+x)\,y = x^3$

or $\frac{d^2y}{dx^2} - 2\left(\frac{1+x}{x}\right)\frac{dy}{dx} + 2\left(\frac{1+x}{x^2}\right)y = x$

Here $P + Qx = -2\left(\frac{1+x}{x}\right) + 2\left(\frac{1+x}{x}\right) = 0$, so $y = x$

is a part of C.F.

13. Given $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$

put $x = e^z$ and $D = \frac{d}{dz}$ we get

$[D(D-1) + D - 9]\;y = 0$

$(D^2 - 9)\;y = 0$ A.E. is $m = \pm 3$

so solution is $y = c_1e^{3z} + c_2e^{-3z}$

or $y = c_1x^3 + \frac{c_2}{x^3}$

16. Given that $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)\;y = e^{x^2}$

Here, $P = -4x$, $Q = 4x^2 - 3$, $R = e^{x^2}$

$u = e^{-1/2\int P dx}$

$= e^{-1/2\int -4x\,dx}$

$= e^{x^2}$

$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}.16x^2 = 1$

and $Y = Re^{\frac{1}{2}\int P dx} = e^{x^2}e^{\frac{1}{2}\int -4x\,dx} = e^{x^2}e^{-x^2} = 1$

So, normal form is $\frac{d^2v}{dx^2} + Xv = Y$

⇒ $\frac{d^2v}{dx^2} + v = 1$

21. Given differential equation is $\sin^2 x\frac{d^2y}{dx^2} = 2y$ and $y = \cot x$ is a solution so put $y = v\cot x$

$\frac{dy}{dx} = \cot x\frac{dv}{dx} - v\,\text{cosec}^2 x$

Again differentiating

$\frac{d^2y}{dx^2} = \cot x\frac{d^2v}{dx^2} - 2\,\text{cosec}^2 x\frac{dv}{dx} + 2\,\text{cosec}^2 x\cot x.v$

Put in given equation, we get $\frac{d^2v}{dx^2} - \frac{2}{\sin x\cos x}\frac{dv}{dx} = 0$

Put $\frac{dp}{dx} = p$ we have $\frac{dp}{p} = \frac{2\sec^2 x}{\tan x}dx$

Integrating we get $p = c_1\tan^2 x$

i.e. $\frac{dv}{dx} = c_1(\sec^2 x - 1)$

Again integrating $v = \frac{y}{\cot x} = c_1\tan x - c_1x + c_2$

or $y = c_1 - c_1x\cot x + c_2\cot x$

23. $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -4x\,dx} = e^{x^2}$

24. Given $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$

Here $P = -\frac{2}{x}$, $Q = 1 + \frac{2}{x^2}$, $R = xe^x$

To convert it in normal form $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$X = 1 + \frac{2}{x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{1}{x^2} = 1$

26. Given $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 0$

Here $P = \tan x$, $Q = -2\cos^2 x$, $R = 0$

Changing the independent variable from x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$$

Choose $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-2\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = \text{Constant} = -2$

so $\frac{dz}{dx} = \cos x$ or $z = \sin x$

so $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = 0$

28. Given $\frac{d^2y}{dx^2} + (\tan x - 1)^2\frac{dy}{dx} - n\,(n-1)\sec^4 xy = 0$

Here $Q = -n\,(n-1)\sec^4 x$, $P = (\tan x - 1)^2$, $R = 0$

To convert it into $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ choose Q such that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-n(n-1)\sec^4 x}{\left(\frac{dz}{dx}\right)^2}$

$$= \text{constant} = -n(n-1)$$

so $\quad \frac{dz}{dx} = \sec^2 x \quad \Rightarrow \quad z = \tan x$

So $\quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$

$$= \frac{2\tan x \sec^2 x + (\tan x - 1)^2 \sec^2 x}{\sec^4 x}$$

or $\quad P_1 = \frac{2\tan x + 1 + \tan^2 x - 2\tan x}{\sec^2 x} = 1$

29. Given $\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6}y = \frac{1}{x^8}$

Here, $P = \frac{3}{x}, Q = \frac{a^2}{x^6}, R = \frac{1}{x^8}$

To change the variable from x to z, we choose

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{a^2}{x^6}}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = a^2$$

so $\quad \frac{dz}{dx} = \frac{1}{x^3}$ i.e. $z = -\frac{1}{2x^3}$

Now transformed equation is

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

where $\quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-3}{x^4} + \frac{3}{x}\cdot\frac{1}{x^3}}{\left(\frac{dz}{dx}\right)^2} = 0$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{1}{x^8}}{\frac{1}{x^6}} = \frac{1}{x^2} = -2z$$

so, we get $\frac{d^2y}{dz^2} + a^2 y = -2z$

$\Rightarrow \quad (D^2 + a^2)\, y = -2z$

30. Since $y = A\cos 2x + B\sin 2x$ be the complete primitive of $(D^2 + 4)\, y = 4\tan 2x$

Here $u = \cos 2x$, $v = \sin 2x$ so Wronskian W is

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

31. Given differential equation is

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = e^x \log x$$

A.E. is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

So. C.F. is $y = c_1e^x + c_2xe^x$

Here, $u = e^x$, $v = xe^x$ so Wronskian W is

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$= e^{2x}(1 + x - x) = e^{2x}$

33. Given $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$

put $x = e^2$ and $D \equiv \frac{d}{dz}$ we have

$[D(D-1) + D - 1]\, y = e^{2z}.e^{e^z}$

$\Rightarrow \quad (D^2 - 1)\, y = e^{2z}.e^{e^z}$

A.E. is $m^2 - 1 = 0, m = 1, -1$

So C.F. is $y = c_1e^z + c_2e^{-z} = c_1x + \frac{c_2}{x}$

So, $u = x, v = \frac{1}{x}$ and Wronskian W is

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = \frac{-2}{x}$$

35. Given $y = vx^3$ be the solution is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$. Differentiating $y = vx^3$ we get

$$\frac{dy}{dx} = x^3\frac{dv}{dx} + 3x^2v$$

Again differentiating $\frac{d^2y}{dx^2} = x^3\frac{d^2v}{dx^2} + 6x^2\frac{dv}{dx} + 6xv$

put these values in given equations we have

$$x^5\frac{d^2v}{dx^2} + 6x^4\frac{dv}{dx} + 6x^3v + x^4\frac{dv}{dx} + 3x^3v - 9x^3v = 0$$

or $\quad x^5\frac{d^2v}{dx^2} + 7x^4\frac{dv}{dx} = 0 \quad \Rightarrow \quad \frac{d^2v}{dx^2} + \frac{7}{x}\frac{dv}{dx} = 0$

i.e. $\quad \left(D^2 + \frac{7}{x}D\right)v = 0$

37. Given $y = v\sec x$

so $\quad \frac{dy}{dx} = \sec x\frac{dv}{dx} + \sec x \tan x v$

again $\dfrac{d^2y}{dx^2} = \sec x \dfrac{d^2v}{dx^2} + 2\sec x \tan x \dfrac{dv}{dx}$

$$+(\sec^3 x + \sec x \tan^2 x)\, v$$

put these values in

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x$$

we have $(D^2 + 6)\, v = e^x$

40. Given $\dfrac{d^2y}{dx^2} + (3\sin x - \cot x)\dfrac{dy}{dx} + 2\sin^2 xy$

$$= e^{-\cos x} \sin^2 x$$

Here, $P = 3\sin x - \cot x$, $Q = 2\sin^2 x$,

$$R = e^{-\cos x} \sin^2 x$$

Choose Q_1 such that

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{2\sin^2 x}{\left(\dfrac{dz}{dx}\right)^2} = \text{Constant} = 2$$

$$\therefore \qquad \frac{dz}{dx} = \sin x$$

i.e., $\qquad z = -\cos x$

So, $\qquad R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{e^{-\cos x}\sin^2 x}{\sin^2 x}$

$$= e^{-\cos x} = e^z$$

42. Given that $[3x^2D^2 + (2 + 6x - 6x^2)\, D - 4]\, y = 0$

or $[(3x^2D^2 + 6xD + 2D) - (6x^2D + 4)]\, y = 0$

or $\qquad [D\,(3x^2D + 2) - 2\,(3x^2D + 2)]\, y = 0$

or $\qquad (D - 2)\,(3x^2D + 2)\, y = 0$

44. Given that $[xD^2 + (x - 1)\, D - 1]\, y = x^2$

or $\qquad [xD(D + 1) - (D + 1)]\, y = x^2$

or $\qquad (xD - 1)\,(D + 1)\, y = x^2$

47. Differentiate $y = A\cos x + B\sin x$ ...(1)

$$\frac{dy}{dx} = -A\sin x + B\cos x + \cos x \frac{dA}{dx} + \sin x \frac{dB}{dx}$$

Choose A and B such that

$$\cos x \frac{dA}{dx} + \sin x \frac{dB}{dx} = 0 \qquad \text{...(2)}$$

Then $\dfrac{dy}{dx} = -A\sin x + B\cos x$

so, $\dfrac{d^2y}{dx^2} = -A\cos x - B\sin x - \sin x \dfrac{dA}{dx} + \cos x \dfrac{dB}{dx}$

Put these value is given differential equation $(D^2 + 1)\, y = x$ we have

$$-A\cos x - B\sin x - \sin x \frac{dA}{dx} + \cos x \frac{dB}{dx}$$

$$+ A\cos x + B\sin x = x$$

i.e. $\qquad -\sin x \dfrac{dA}{dx} + \cos x \dfrac{dB}{dx} = x \qquad$ ...(3)

Multiplying (2) by $\cos x$ and (3) by $\sin x$ and Subtracting we get

$$\frac{dA}{dx} = -x\sin x$$

Integrating $A = x\cos x - \sin x + c$

50. The complete primitive is $y = Ax + Bxe^{2x}$

Differentiate w.r.t.x,

$$\frac{dy}{dx} = A + B\,(e^{2x} + 2xe^{2x}) + x\frac{dA}{dx} + xe^{2x}$$

Choose A and B such that

$$x\frac{dA}{dx} + xe^{2x}\frac{dB}{dx} = 0 \qquad \text{...(1)}$$

Then $\dfrac{dy}{dx} = A + B\,(1 + 2x)\, e^{2x}$

and $\dfrac{d^2y}{dx^2} = \dfrac{dA}{dx} + \dfrac{dB}{dx}e^{2x}(1 + 2x)$

$$+ 2Be^{2x} + 2B\,(1 + 2x)\, e^{2x}$$

Put these values in given equation

$$\frac{d^2y}{dx^2} - \frac{2\,(1 + x)}{x}\frac{dy}{dx} + \frac{2\,(x + 1)}{x^2}y = x$$

We get, $\dfrac{dA}{dx} + e^{2x}(1 + 2x)\dfrac{dB}{dx} = x \qquad$ ...(2)

Solving (1) and (2), we get $\dfrac{dA}{dx} = -\dfrac{1}{2}$

$$\Rightarrow \qquad A = -\frac{1}{2}x + c$$

52. Given equation is $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \left(1 + \dfrac{2}{x^2}\right)y = 0$

Here $\qquad P = -\dfrac{2}{x}$, $Q = 1 + \dfrac{2}{x^2}$, $R = 0$

Choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x}dx} = x$

Put $y = uv$ in given equation it reduces to normal for

$$\frac{d^2v}{dx^2} + Xv = Y$$

Where $X = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{P^2}{4} = 1 + \dfrac{2}{x^2}$

$$-\frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{1}{4}.\frac{4}{x^2} = 1$$

and $\qquad R = Re^{\frac{1}{2}\int P dx} = 0$, so we get

$$\frac{d^2v}{dx^2} + v = 0 \quad \Rightarrow \quad (D^2 + 1)\, v = 0$$

54. Given $y = A\cos x + B\sin x$ be the solution of $\dfrac{d^2y}{dx^2} + y = x$, then

$$\frac{dy}{dx} = -A\sin x + B\cos x + \cos x\frac{dA}{dx} + \sin x\frac{dB}{dx}$$

Choose A and B such that

$$\cos x\frac{dA}{dx} + \sin x\frac{dB}{dx} = 0 \qquad ...(1)$$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -A\cos x - B\sin x - \frac{dA}{dx}\sin x + \frac{dB}{dx}\cos x$$

Put these values in given differential equation, we get

$$-\sin x\frac{dA}{dx} + \cos x\frac{dB}{dx} = x \qquad ...(2)$$

Solving (1) and (2) we have $\dfrac{dB}{dx} = x\cos x$

So, $\quad B = x\sin x + \cos x + c$

56. Given $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + n^2y = 0$

Here, $\quad P = \dfrac{2}{x}, Q = n^2, R = 0$

$$\text{Choose } u = e^{-\frac{1}{2}\int P dx} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Put $y = uv$ in given equation, it reduces to its normal form

$$\frac{d^2v}{dx^2} + Xv = Y$$

$$\text{Where } X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = n^2$$

$$-\frac{1}{2}\left(-\frac{1}{x^2}\right) - \frac{1}{4}.\frac{4}{x^2} = n^2$$

58. If $y = e^x$ is a part of C.F. of $(D^2 - \cot xD - Q)\, y = e^x \sin x$ then $1 + P + Q = 0$

$\Rightarrow 1 - \cot x - Q = 0 \quad \Rightarrow \quad Q = 1 - \cot x$

60. Given $y = Ax + \dfrac{B}{x}$ be the general solution of

$$(x^2D^2 + xD - 1)\, y = x^2e^x \qquad ...(1)$$

Differentiating we get

$$\frac{dy}{dx} = A - \frac{B}{x^2} + x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx}$$

Choose A and B such that

$$x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} = 0 \qquad ...(2)$$

$$\text{then} \qquad \frac{dy}{dx} = A - \frac{B}{x^2}$$

$$\text{and} \qquad \frac{d^2y}{dx^2} = \frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} + \frac{2}{x^3}B$$

Putting these values in equation (1) we get

$$\frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} = e^x \qquad ...(3)$$

Solving (2) and (3) we have $\dfrac{dA}{dx} = \dfrac{1}{2}e^x$

Integrating $A = \dfrac{1}{2}e^x + c$

63. Given $\dfrac{d^2y}{dx^2} + 2\tan x\dfrac{dy}{dx} + (1 + 2\tan^2 x)\, y$

$$= \sec x \tan x$$

Here, $P = 2\tan x$, $Q = 1 + 2\tan^2 x$, $R = \sec x \tan x$

$$Y = Re^{\frac{1}{2}\int P dx} = \sec x \tan x e^{\frac{1}{2}\int 2\tan x dx}$$

$$Y = \sec x \tan x . e^{\log \sec x} = \sec^2 x \tan x$$

65. Given $\left(D^2 - \dfrac{1}{x}D + 4x^2\right)y = x^4$

Here, $\qquad P = -\dfrac{1}{x}, Q = 4x^2, R = x^4$

$$\text{Choose } Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1$$

$$\frac{dz}{dx} = 2x \quad \Rightarrow \quad z = x^2$$

Put $z = x^2$, now equation is

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$$

$$\text{Where} \qquad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(-\frac{1}{x}\right)2x}{4x^2} = 0$$

66. Given $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - y\sin^2 x = 0$

$$\text{Choose } Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sin^2 x}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = -1$$

So, $\frac{dz}{dx} = \sin x, z = -\cos x$ given equation reduces to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R, \text{ where}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cot x.\sin x}{\sin^2 x} = 0$$

70. $(D^2 + 3D + 2)\,x = 5$

A.E. is $m^2 + 3m + 2 = 0$

$$\Rightarrow \quad m = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$\therefore \quad m = -1, -2$$

so C.F. is $c_1e^{-t} + c_2e^{-2t}$

$$\text{P.I. } \frac{1}{(D^2 + 3D + 2)}.5 = \frac{1}{2}\left(1 + \frac{3D}{2} + \frac{D^2}{2}\right)^{-1}.5 = \frac{5}{2}$$

$\therefore$ Solution is $x = c_1e^{-t} + c_2e^{-2t} + \frac{5}{2}$

so, $\lim\limits_{t \to \infty} x = \frac{5}{2}$

71. Given equation is $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = 0$

Here, $P = -\frac{2}{x}, Q = 1 + \frac{2}{x^2},$

So, $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$$= 1 + \frac{2}{x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{4}{4x^2} = 1$$

* * *

Unit-II

CHAPTER 8

Partial Differential Equations of the First Order

PARTIAL DIFFERENTIAL EQUATIONS

1. Equations which contains one or more partial derivatives are called partial differential equations. Thus the partial differential equation contains at least two independent variables.

   We shall denote $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r$

   $$\frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

2. The order of a partial differential equation is the order of the highest order derivative occurs in the differential equation.

3. The degree of a partial differential equation is the degree of the highest order derivative which occur in it after the differential equation has been rationalised *i.e.* made free form redicals and fractions so far as derivatives are concerned.

DERIVATION OF A PARTIAL DIFFERENTIAL EQUATION

The partial differential equation can be formed by two methods

1. Elimination of arbitrary constants
   Consider the equation

   $$f(x, y, z, a, b) = 0 \qquad ...(1)$$

   Where a, b are arbitrary constants and let z be the function of x and y

   Differentiating equation (1) partially w.r.t. x and y respectively we get

   $$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad ...(2)$$

   and $$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \qquad ...(3)$$

   Eliminating two constants a and b from equations, (1), (2) and (3) we obtained

   $$f(x, y, z, p, q) = 0$$

   which is the required partial differential equations of first order.

2. Elimination of Arbitrary Functions
   Consider $f(u, v) = 0 \qquad ...(1)$

   where u and v are functions of x, y and z.

   Differentiating (1) partially w.r.t. independent variables x and y regarding z as dependent variable, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad ...(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between(2) and (3) we get

$$p\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right) + q\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right) = \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}$$

i.e. $Pp + Qq = R \qquad ...(4)$

where, $$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}$$

$$Q = \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

and $$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Equation (4) is the required partial differential equation.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

1. A first order partial differential equation

   $f(x, y, z, p, q) = 0$ in which dependent variable is z which is a function of two independent variables x and y, is called a linear if the dependent variable z and its partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ all occur in first degree.

2. A first order partial differential equation in which dependent variable is z which is a function of two independent variables x and y, is called a non-linear equation if p and q do not occur in it first degree.

3. A first order partial differential equation in which dependent variable is z which is a function of two independent variables x and y, is called a quasi-linear equation if f is a linear expression in p and q but not necessarily linear in z.

4. A first order partial differential equation in which dependent variable is z which is a function of two independent variables x and y is called a semi-linear equation if two coefficient of p and q are function of x and y only and they do not depend on z. The terms that do not involve p and q contains the terms that are not of first degree in z.

LAGRANGE'S LINEAR PARTIAL DIFFERENTIAL EQUATION

1. The partial differential equation $Pp + Qq = R$, where P, Q and R are functions of x, y and z is called Lagrange's linear partial differential equation of first order.

2. Consider Lagrange's partial differential equation

$$Pp + Qq = R \qquad \text{...(1)}$$

Let $u = a$ be an integral of (1)

Differentiating partially it w.r.t. x and y we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0 \text{ and } \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0$$

i.e.
$$p = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}},\ q = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}$$

Put these values in (1) we get

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \qquad \text{...(2)}$$

Thus if $u = a$ satisfies (1) it also satisfies (2).

Conversely if $u = a$ is in integral of (2) then it is also an integral of (1). Hence (1) and (2) are equivalent.

3. The general solution of the linear partial differential equation

$$Pp + Qq = R \qquad \text{...(1)}$$

If $f(u, v) = 0$ where f is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

4. If the linear equation with n independent variables $x_1, x_2, \ldots, x_n$ be

$$P_1p_1 + P_2p_2 + \ldots . P_np_n = R \qquad \text{...(1)}$$

Where $P_1, P_2, \ldots., P_n$ and R are functions of $x_1, x_2, \ldots., x_n$ and z. Then the general solution of (1) is given by $f(u_1, u_2, \ldots, u_n) = 0$

Where $u_i(x_1, x_2, \ldots, x_n, z) = c_i$, $i = 1, 2, \ldots, n$ are any independent integrals of the auxiliary equations $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots \frac{dx_4}{P_n} = \frac{dz}{R}$

5. The general solution of $Pp + Qq = R$ is the family of surfaces such that the normal to a surface at any point must be perpendicular to the tangent to a curve of the family of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

6. The equation $Pp + Qq = R$ represents a family of surfaces orthogonal to the family of surfaces represented by $Pdx + Qdy + Rdz = 0$.

7. For the solution of Lagrange's equation $Pp + Qq = R$ form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Find two independent integrals say $u = a$ and $v = b$ then the general solution of the partial differential equation $Pp + Qq = R$ is given by $f(u, v) = 0$, when f is an arbitrary function.

THE INTEGRALS OF THE NON-LINEAR EQUATIONS

1. Any relation which contains as many arbitrary constants as there are independent variables and is a solution of a partial differential equation of the first order is called a complete solution or a complete integral of that equation.

If $$f(x, y, z, a, b) = 0 \quad ...(1)$$
be the solution of $f(x, y, z, p, q) = 0$ ...(2)

Then (1) is complete solution of complete integral of (2).

2. A particular integral of (2) can be obtained by given particular values to a and b in (1).

3. The equation of the envelope of the surfaces represented by (1) can be obtained by eliminating a and b between

$$f = 0, \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0$$

This equation of the envelope is called the singular integral.

CHARPIT'S METHOD

1. Charpit's method is the general method for solving partial differential equations of order one but of any degree.

Consider the differential equation

$$f(x, y, z, p, q) = 0 \quad ...(1)$$

with $dz = pdx + qdy$ ...(2)

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{\partial f}{0} \quad ...(3)$$

Any two of equation (3) give a relation of p and q which is solved for p, q with equation (1). Substitute these values in equation (2) and then integrating it, we get the complete solution of the given partial differential equation.

SPECIAL METHODS OF SOLUTION APPLICABLE TO CERTAIN STANDARD FORMS

1. Equations involving only p and q and not x, y, z.

Let the equation is $f(p, q) = 0$ ...(1)

The complete integral is $z = ax + by + c$ ...(2)

Where a and b are connected by

$$f(a, b) = 0 \quad ...(3)$$

By (2),

$$p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b$$

Put $p = a$ and $q = b$ in (3), we get equation (1)

By equation (3), let $b = \phi(a)$

So complete integral of (1) is

$$z = ax + \phi(a).y + c \quad ...(4)$$

General Integral :

Put $c = \psi(a)$ in (4) we get

$$z = ax + \phi(a)\,y + \psi(a) \quad ...(5)$$

Differentiating it partially w.r.t. a we get

$$0 = x + \phi'(a)y + \psi'(a) \quad ...(6)$$

The general integral is obtained by eliminating a between (5) and (6).

Singular Integral : The singular solution (integral) is obtained by eliminating a and c between the complete integral (4) and the equations formed by differentiating (4) partially w.r.t. a and c *i.e.*

$$z = ax + \phi(a)\,y + c$$
$$0 = x + \phi'(a)\,y$$
$$0 = 1$$

which is not exist so there is no singular solution.

2. Equation involving only p, q and z *i.e.*

$$f(p, q, z) = 0 \quad ...(1)$$

Assume that $z = f(x + ay)$ be the trial solution of (1) then

$$z = f(X) \text{ where } X = x + ay$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{dX}\frac{\partial X}{\partial x} = \frac{dz}{dX} \quad \because \quad \frac{\partial X}{\partial x} = 1$$

and $$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}.\frac{\partial X}{\partial y} = a\frac{dz}{dX}$$

So, equation (1) reduces to $f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0$

Which is an ordinary differerntial equation of order one. Integrating it we get the complete integral.

3. Equation of the form $f(x, p) = \phi(y, q)$.

Consider $f(x, p) = \phi(y, q) = a$ ...(1)

Then we get $p = f_1(x, a)$, $q = f_2(y, a)$

We know $dz = pdx + qdy$

so, $$dz = f_1(x, a)\,dx + f_2(y, a)\,dy$$

Integrating we get

$$z = \int f_1(x, a)\,dx + \int f_2(y, a)\,dy + b$$

Which is the complete integral. There is no any singular integral.

4. Equation of the form

$$z = px + qy + f(p, q) \qquad ...(1)$$

The complete integral of Clairaut's equation (1) is

$$z = ax + by + f(a, b)$$

$$\therefore \quad \frac{\partial F}{\partial a} = 0 \Rightarrow x + \frac{\partial f}{\partial a} = 0 \qquad ...(2)$$

$$\text{and} \quad \frac{\partial F}{\partial b} = 0 \Rightarrow y + \frac{\partial f}{\partial b} = 0 \qquad ...(3)$$

Singular integral is obtained by eliminating a and b from (1), (2) and (3).

COMPATIBLE SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

1. If every solution of the first order partial differential equation

$$f(x, y, z, p, q) = 0 \qquad ...(1)$$

is also a solution of the first order partial differential equation

$$\phi(x, y, z, p, q) = 0 \qquad ...(2)$$

Then the differential equations (1) and (2) are said to be compatible.

2. The condition that the equation (1) and (2) are compatible is

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

3. The first order partial differential equations $p = P(x, y)$, $q = Q(x, y)$ are compatible if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The partial differential equation of $z = (x + a)(y + b)$ is :

(a) $z = p + q$ (b) $z = px + qy$
(c) $z = xyp$ (d) $z = pq$

2. The partial differential equation of $z = a(x + y) + b$ is :

(a) $pq = 0$ (b) $p = q$
(c) $p + q = 0$ (d) $p - q = 0$

3. The order of $r - 2s + p^3 = 0$ is :

(a) 1 (b) 2 (c) 3 (d) 0

4. The degree of $r^3 + 2s + t^2 = 0$ is :

(a) 1 (b) 2
(c) 3 (d) None of these

5. The differential equation $zyp - xyq = x(z - 2y)$ is :

(a) Linear (b) Quasi-linear
(c) Semi-linear (d) Non-linear

6. The differential equation $pq = z$ is :

(a) Linear (b) Non-linear
(c) Semi-linear (d) Quasi linear

7. The differential equation $x^2p + (y^3 + x^3)q = x^2 + y^2 + z^2$ is :

(a) Linear (b) Semi-linear
(c) Non-linear (d) Quasi-linear

8. The differerntial equation of $z = f\left(\frac{y}{x}\right)$ is :

(a) $px + qy = 0$ (b) $pq + xy = 0$
(c) $py + qx = 0$ (d) $pq = x + y$

9. The differential equation of $z = (x^2 + a)(y^2 + b)$ is :

(a) $pq = xyz$ (b) $q = px + y$
(c) $p = qy + z$ (d) $pq = 4xyz$

10. The differential equation of $z = f(x^2 + y^2)$ is :

(a) $xp = yq$ (b) $xp^2 = yq^2$
(c) $yp^2 = xq^2$ (d) $yp = xq$

11. The auxiliary equation of $Pp + Qq = R$ is :

(a) $Pdx + Qdy + Rdz = 0$
(b) $Pdp + Qdq + Rdz = 0$
(c) $\frac{dp}{P} = \frac{dq}{Q} = \frac{dz}{R}$ (d) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

12. The order of the differerntial equation $\frac{\partial^2 z}{\partial x^2} + u^2\frac{\partial^2 x}{\partial y^2} = f(x, y)$ is :

(a) 1 (b) 2
(c) 3 (d) None of these

13. The solution of $z(xp - yq) = y^2 - x^2$ is :

(a) $f\left(\frac{x}{y}, x^2 + y^2 + z^2\right) = 0$
(b) $f(xyz, x^2 + y^2 + z^2) = 0$
(c) $f(yz, x^2 + y^2 + z^2) = 0$
(d) $f(xy, x^2 + y^2 + z^2) = 0$

14. The solution of $\left(\frac{y-z}{yz}\right)p+\left(\frac{z-x}{zx}\right)q=\frac{x-y}{xy}$ is :

(a) $f\,(x+y+z,\ xyz)=0$

(b) $f\,(x-y-z,\ xy)=0$

(c) $f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0$

(d) $f\left(\frac{1}{x+y+z},\ xyz\right)=0$

15. The solution of $p+q=\frac{z}{a}$ is :

(a) $z=e^{\frac{a}{y}}f(x+y)$ (b) $z=e^{\frac{y}{a}}f(x-y)$

(c) $z=e^{-y}f(x-y)$ (d) $z=e^{y}f(x-y)$

16. The solution of $p\tan x+q\tan y=\tan z$ is :

(a) $\frac{\sin y}{\sin z}=f\left(\frac{\sin x}{\sin z}\right)$ (b) $\frac{\sin z}{\sin x}=f\left(\frac{\sin x}{\sin y}\right)$

(c) $\frac{\sin z}{\sin y}=f\left(\frac{\sin x}{\sin y}\right)$ (d) $\frac{\sin z}{\sin y}=f\left(\frac{\sin z}{\sin y}\right)$

17. The solution $zy^2p-zx^2q=x^2y$ is :

(a) $f\,(x^3+y^3,\ y^2+z^2)=0$

(b) $f\,(x^3+y^3+z^3,\ xyz)=0$

(c) $f\,(x^2+y^2+z^2,\ xyz)=0$

(d) $f\,(x+y+z,\ xyz)=0$

18. The complete integral of $3p^2=q$ is :

(a) $z=ax+ay+b$ (b) $z=ax+3a^2y+b$

(c) $z=ax+\frac{a}{y}+b$ (d) $z=ax+a^2y+b$

19. The complete integral of $xp+yq=z$ is :

(a) $f(xy,\ yz)=0$ (b) $f\,(x,\ y)=0$

(c) $f\,(xy,\ z)=0$ (d) $f\left(\frac{x}{y},\ \frac{y}{z}\right)=0$

20. The singular integral of $p^2+q^2=1$ is :

(a) $z=0$ (b) $x=0$

(c) $y=0$ (d) Not exist

21. The solution of $p^2=zq$ is :

(a) $z=be^{x^2+ay}$ (b) $z=be^{ax+xy^2}$

(c) $z=be^{ax+a^2y}$ (d) $z=be^{axy+b}$

22. The complete integral of $z=px+qy+p^2+q^2$ is :

(a) $z=ax+by+a^2+b^2$ (b) $z=px+qy+a^2+b^2$

(c) $z=ax+by+ab$ (d) $z=ax+by+a+b$

23. The solution of $px^2+qy^2=z^2$ is :

(a) $\frac{1}{y}-\frac{1}{z}=f\left(\frac{1}{x}-\frac{1}{y}\right)$ (b) $y-z=f\,(x-y)$

(c) $\frac{1}{y}+\frac{1}{z}=f\left(\frac{1}{x}+\frac{1}{y}\right)$ (d) $y+z=f\,(x+y)$

24. The solution of $z(px+qy)=y^2-x^2$ is :

(a) $f(xyz,\ x^2+y^2+z^2)=0$

(b) $f\,(xyz,\ x+y+z)=0$

(c) $f\,(xy,\ x+y+z)=0$

(d) $f\left(\frac{x}{y},\ -x^2-y^2+z^2\right)=0$

25. The solution of $-p_1+p_2+p_3=1$ is :

(a) $f\,(x_1+x_2,\ x_1+x_3,\ x_1+y)=0$

(b) $f\,(x_1+x_2, x_1+x_3, x_1+z)=0$

(c) $f\,(x_1+x_2,\ x_1+x_3,\ x_1+x)=0$

(d) $f\,(x_1+x_2,\ x_2+x_3,\ x_3+x_1)=0$

26. The family of surfaces orthogonal to the family of surfaces of $(y+z)\,p+(z+x)\,q=x+y$ is :

(a) $xyz+c=0$ (b) $xy+yz+zx=c$

(c) $x+y+z=0$ (d) $x^2+y^2+z^2=0$

27. The partial differential equation from $z=(a+x)^2+y$ is :

(a) $z=\frac{p}{2}+y$ (b) $z=\frac{p^2}{4}+y$

(c) $z=\frac{q^2}{4}+y$ (d) $z=\frac{q}{2}+y$

28. The complete integral of $q=e^{-\frac{p}{a}}$ is :

(a) $z=ye^{-\frac{b}{a}}+c$ (b) $z=bx+ye^{\frac{b}{a}}+c$

(c) $z=bx+ye^{-\frac{b}{a}}+c$ (d) $z=ye^{-\frac{b}{a}x}+c$

29. The complete integral of $\sqrt{p}+\sqrt{q}=1$ is :

(a) $z=ax+(1-\sqrt{a})^2y+b$

(b) $z=ax+\sqrt{a}y+b$

(c) $z=ax+(1+\sqrt{a})^2y$

(d) $z=ax+(1-\sqrt{a})\ y+b$

30. The complete integral of $p\,(1+q)=qz$ is :

(a) $az=1+be^{ax+y}$ (b) $az=be^{ax+y}$

(c) $az=1+be^{x+ay}$ (d) $az=be^{x+ay}$

31. The differential equations $p=4x+3y+1$ and $q=3x+2y+1$ are :

(a) Exact (b) Compatible

(c) Not compatible (d) Non-linear

32. The equations $f(x,y,p,q) = 0$ and $g(x,y,p,q) = 0$ are compatible if :

(a) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$
(b) $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$
(c) $\frac{\partial (f, g)}{\partial (x, p)} + \frac{\partial (f, g)}{\partial (y, q)} = 0$
(d) $\frac{\partial (f, g)}{\partial (x, p)} = \frac{\partial (f, g)}{\partial (y, q)}$

33. The differential equations $p = 7x + 8y - 1$ and $q = 3x + 5y - 6$ are :

(a) Exact
(b) Compatible
(c) Non-linear
(d) Not compatible

34. The equation $Pp = Qq = R$ is called :

(a) Charpit's equation
(b) Bernoulli's equation
(c) Lagrange's equation
(d) Monge's equation

35. The differential equations $p = 2x - y$, $q = 5 - x - 2y$ are :

(a) Compatible
(b) Non-compatible
(c) Exact
(d) Non-linear

36. The singular integral of $z = px + qy + \log pq$ is :

(a) $z = 2 + \log xy$
(b) $z = qx + \log y$
(c) $z = qy + \log x$
(d) $z = -2 - \log xy$

37. The singular integral of $z = px + qy + p^2 + q^2$ is :

(a) $x^2 + y^2$
(b) $-(x^2 + y^2)$
(c) $-\left(\frac{x^2 + y^2}{2}\right)$
(d) $-\left(\frac{x^2 + y^2}{4}\right)$

38. The singular integral of $z = px + qy - p^2 q$ is :

(a) $z = \frac{x\sqrt{y}}{2}$
(b) $z = x\sqrt{y}$
(c) $z = \frac{\sqrt{xy}}{2}$
(d) $z = \sqrt{xy}$

39. The differential equation of $z = ax + a^2 y^2 + b$ is :

(a) $q = 2yp^2$
(b) $p = 2xq^2$
(c) $q = 2xyp$
(d) $p = 2xyq$

40. The differential equation of $z = Ae^{pt} \sin px$ is :

(a) $\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial t^2} = 0$
(b) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
(c) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
(d) $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$

41. The differential equation $(x + 2y)\, p + xp - pq = x + y$ is :

(a) Linear
(b) Semi-linear
(c) Quasi-linear
(d) Non-linear

42. The differential equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible iff :

(a) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$
(b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
(c) $\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial y^2}$
(d) $\frac{\partial^2 P}{\partial y^2} = \frac{\partial^2 P}{\partial x^2}$

43. The partial differential equation of $z = f(x + ay) + \phi(x - ay)$ is :

(a) $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$
(b) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = a^2$
(c) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = a^2$
(d) $a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$

44. The Charpit's auxiliaries equations of $p(1 + q^2) = q(z - b)$ is :

(a) $\frac{dp}{p^2} = \frac{dq}{q^2}$
(b) $\frac{dp}{q} = \frac{dq}{p}$
(c) $\frac{dp}{zp} = \frac{dq}{bz}$
(d) $\frac{dp}{z} = \frac{dq}{pq}$

45. The Charpit's auxiliary equation of $px + qy = pq$ is :

(a) $\frac{dx}{x + p} = \frac{dy}{y + q}$
(b) $\frac{dx}{x - p} = \frac{dy}{y - q}$
(c) $\frac{dx}{q + x} = \frac{dy}{p + y}$
(d) $\frac{dx}{q - x} = \frac{dy}{p - y}$

46. The Charpit's auxiliary equation of $-px^2 + 2zx - 2qxy + pq = 0$ is :

(a) $\frac{dp}{0} = \frac{dx}{x^2 - q}$
(b) $\frac{dq}{0} = \frac{dx}{x^2 - q}$
(c) $\frac{dq}{0} = \frac{dx}{x^2 - q}$
(d) $\frac{dq}{0} = \frac{dx}{x}$

47. The Charpit's auxiliary equation of $p^2 + q^2 - 2px - 2qy + 1 = 0$ is :

(a) $\frac{dp}{x} = \frac{dq}{y}$
(b) $\frac{dp}{p} = \frac{dq}{q}$
(c) $\frac{dp}{q} = \frac{dq}{p}$
(d) $\frac{dp}{y} = \frac{dq}{x}$

48. The Charpit's auxiliary equation of $p = (qy + z)^2$ is :

(a) $\frac{dp}{p} = \frac{dq}{q}$
(b) $\frac{dp}{p} = \frac{dx}{x}$
(c) $\frac{dp}{p} = \frac{dy}{y}$
(d) $\frac{dp}{p} = \frac{dy}{-y}$

49. The complete integral of $p^2 - 3q^2 = 5$ is :

(a) $z = \sqrt{5 - 3a^2}\, x + ay + b$
(b) $z = \sqrt{5}\, x + ay + b$
(c) $z = \sqrt{3a^2}\, x + ay + b$
(d) $z = \sqrt{5 + 3a^2}\, x + ay + b$

50. The complete integral of $p^2q^2(px + qy - z) = 2$ is :

(a) $z = ax + y - 2a^2b^2$ (b) $z = ax + by - \dfrac{2}{a^2b^2}$

(c) $z = ax + by + 2ab$ (d) $z = x + y - \dfrac{2}{a^2b^2}$

51. The Charpit's equation of $(p^2 + q^2)\,x = pz$ is :

(a) $\dfrac{dp}{-q^2} = \dfrac{dq}{pq}$ (b) $\dfrac{dp}{p^2} = \dfrac{dq}{pq}$

(c) $\dfrac{dp}{p} = \dfrac{dq}{q^2}$ (d) $\dfrac{dp}{q^2} = \dfrac{dq}{p^2}$

52. The Charpit's equation of $px + qy + z - xq^2 = 0$ is :

(a) $\dfrac{dx}{x} = \dfrac{dq}{q}$ (b) $\dfrac{dx}{2x} = \dfrac{-dq}{q}$

(c) $\dfrac{dx}{x^2} = \dfrac{dq}{q^2}$ (d) $\dfrac{dx}{x} = -\dfrac{dq}{2q}$

53. The Charpit's auxiliary equation of $p^2x + q^2y = z$ is :

(a) $\dfrac{dp}{p^2 - p} = \dfrac{dq}{q^2 - q} = \dfrac{dx}{-2px}$

(b) $\dfrac{dp}{p^2 + p} = \dfrac{dq}{q^2 - q} = \dfrac{dx}{-1px}$

(c) $\dfrac{dp}{p^2 - p} = \dfrac{dq}{q^2 + q} = \dfrac{dx}{-2px}$

(d) $\dfrac{dp}{p^2 - p} = \dfrac{dq}{q^2 - q} = \dfrac{dx}{2px}$

54. The complete integral of $py = 2xy + \log q$ is :

(a) $z = x^2 + ax + \log y + b$

(b) $z = 2xy + \log b$

(c) $z = x^2 + ax + \dfrac{e^{ay}}{a} + b$

(d) $z = x^2 + x + e^{ay} + b$

55. The complete integral of $z = px + qy + \dfrac{q}{p} - p$ is :

(a) $z = ax + by + a - b$

(b) $az = a\,(ax + by) + b - a^2$

(c) $z = ax + by + \dfrac{b^2 - a^2}{a}$

(d) $z = ax + by + a + b$

56. The complete integral of $pq = xy$ is :

(a) $az = x^2 + y^2 + 2ab$

(b) $2az = ax^2 + by^2$

(c) $2az = a^2x^2 + y^2 + 2ab$

(d) $az = ax^2 + by^2 + ab$

57. The Charpit's auxiliary equations of $p^2 + q^2 - 2px - 2qy + 2xy = 0$ are :

(a) $\dfrac{dp}{y - p} = \dfrac{dq}{q - x} = \dfrac{dx}{p - x}$

(b) $\dfrac{dp}{x - p} = \dfrac{dq}{y - q} = \dfrac{dx}{p + x}$

(c) $\dfrac{dp}{y + p} = \dfrac{dq}{q + x} = \dfrac{dx}{p + x}$

(d) $\dfrac{dp}{x + p} = \dfrac{dq}{y + q} = \dfrac{dx}{p - x}$

58. The complete integral of $pe^y = qe^x$ is :

(a) $z = ae^x + be^y$

(b) $az = e^x + be^y$

(c) $bz = ae^x + e^y$

(d) $z = a\,(e^x + e^y) + b$

59. Lagrange's auxiliary equations of $z\,(xp - yq) = y^2 - x^2$ are :

(a) $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dz}{y^2 - x^2}$

(b) $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{y^2 - x^2}$

(c) $\dfrac{dx}{xz} = \dfrac{dy}{-yz} = \dfrac{dz}{y^2 - x^2}$

(d) $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{y^2 - x^2}$

60. Lagrange's auxiliary equations of $Pp + Qq = R$ are :

(a) $\dfrac{dx}{P} = \dfrac{dy}{Q} = dz$ (b) $\dfrac{dx}{Q} = \dfrac{dy}{P} = \dfrac{dz}{R}$

(c) $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ (d) $\dfrac{dx}{R} = \dfrac{dy}{Q} = \dfrac{dz}{P}$

61. The degree of the equation $p^4 + q^2x^6 + z^6 = 0$ is :

(a) 6 (b) 4

(c) 2 (d) 1

62. The order of the equation $r^2 - 2s + t^3 = 0$ is :

(a) 3 (b) 2

(c) 1 (d) None of these

63. If in the Lagrange's auxiliary equations of $(x + 2z)\,p + (4zx - y)\,q = 2x^2 + y$, whose $y, x, -2z$ as multipliers then :

(a) $x^2 - y - z = c$ (b) $x^2 - yz = c$

(c) $xz - y^2 = c$ (d) $xy - z^2 = c$

64. If in the Lagrange's auxiliary equations of $x(y^2+z)\,p - y(x^2+z)\,q = z(x^2-y^2)$, choose $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier then :

(a) $x+y+z=c$
(b) $xyz=c$
(c) $\frac{xy}{z}=c$
(d) $\frac{xz}{y}=c$

65. If in the Lagrange's auxiliary equations of $2y(z-3)\,p + (2x-z)\,q = y(2x-3)$, choose $\frac{1}{z}, y, -1$ as multipliers then :

(a) $x+y^2-2z=c$
(b) $y^2+x-2z=c$
(c) $z+x^2-2y=c$
(d) $y+z^2-2x=c$

66. The Charpit's auxiliary equations of $px+qy = z(1+pq)^{1/2}$ are :

(a) $\frac{dp}{p-q}=\frac{dq}{p+q}$
(b) $\frac{dp}{p^2}=\frac{dq}{q^2}$
(c) $\frac{dp}{p}=\frac{dq}{q}$
(d) $\frac{dp}{p^2-q^2}=\frac{dq}{p^2+q^2}$

67. The Charpit's auxiliary equations of $z^2=pqxy$ are :

(a) $\frac{dx}{q}=\frac{dy}{p}$
(b) $\frac{dx}{px}=\frac{dy}{qy}$
(c) $\frac{dx}{x}=\frac{dy}{y}$
(d) $\frac{dx}{p}=\frac{dy}{q}$

68. The Charpit's auxiliary equations of $p^2+q^2-2px-2qy+2xy=0$ are :

(a) $\frac{dx}{x+q}=\frac{dy}{y+q}$
(b) $\frac{dx}{x-q}=\frac{dy}{y-q}$
(c) $\frac{dx}{x^2}=\frac{dy}{y^2}$
(d) $\frac{dx}{q}=\frac{dy}{p}$

69. In a linear partial differential equation all the partial derivatives occuring in it are of degree :

(a) 1
(b) 2
(c) 3
(d) Any finite value

70. The order of the partial differential equation of $z=\phi(x+iy)+\psi(x-iy)$ is :

(a) 1
(b) 2
(c) 3
(d) 4

71. The degree of the partial differential equation of $z=ax+a^2y^2+b$ is :

(a) 4
(b) 3
(c) 2
(d) 1

72. The solution of $zp+x=0$ is :

(a) $x^2+y^2=f(z)$
(b) $y^2+z^2=f(x)$
(c) $x^2+y^2+z^2=f(xyz)$
(d) $x^2+z^2=f(y)$

73. The Charpit's auxiliary equation of $z=px+qy+p^2+q^2$ is :

(a) $\frac{dp}{p^2+p}=\frac{dq}{q^2+q}$
(b) $\frac{dp}{p}=\frac{dq}{q}$
(c) $pdp=qdq$
(d) $\frac{dp}{0}=\frac{dq}{0}$

74. The singular integral of $p+q=1$ is :

(a) $x=0$
(b) $y=0$
(c) $x+y=0$
(d) Not exist

75. The partial differential equation of $az=xy$ is :

(a) $px=z$
(b) $qx=z$
(c) $py=z$
(d) $px=qy$

76. The differential equation of $z=f(xy)$ is :

(a) $p=q$
(b) $p+q=x+y$
(c) $py=qx$
(d) $px=qy$

77. The differential equation of $z=f(x-y)$ is :

(a) $p+q=0$
(b) $p=q$
(c) $p=x$
(d) $q=y$

78. The differential equation of $z=y^2+2f\left(\frac{1}{x}+\log y\right)$ is :

(a) $px+qy=2y^2$
(b) $px-qy=2y^2$
(c) $p^2x+qy=y^2$
(d) $px^2+qy=2y^2$

79. The differential equation $(x^2-yz)\,p+(y^2-2x)\,q=z^2-xy$ is :

(a) Quasi-linear
(b) Non-linear
(c) Semi-linear
(d) Linear

80. The differential equation $z=axe^y+\frac{1}{2}a^2e^{2y}+b$ is :

(a) $px=z^2$
(b) $q=px+pq$
(c) $q=px+p^2$
(d) $p=2x+q^2$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(b)	**3.**	(b)	**4.**	(c)	**5.**	(b)	**6.**	(b)	**7.**	(b)	**8.**	(a)	**9.**	(d)	**10.**	(d)
11.	(d)	**12.**	(b)	**13.**	(d)	**14.**	(a)	**15.**	(b)	**16.**	(c)	**17.**	(a)	**18.**	(b)	**19.**	(d)	**20.**	(d)
21.	(c)	**22.**	(a)	**23.**	(a)	**24.**	(d)	**25.**	(b)	**26.**	(b)	**27.**	(b)	**28.**	(c)	**29.**	(a)	**30.**	(c)
31.	(b)	**32.**	(c)	**33.**	(d)	**34.**	(c)	**35.**	(a)	**36.**	(d)	**37.**	(d)	**38.**	(b)	**39.**	(a)	**40.**	(b)
41.	(d)	**42.**	(b)	**43.**	(d)	**44.**	(b)	**45.**	(d)	**46.**	(b)	**47.**	(b)	**48.**	(d)	**49.**	(d)	**50.**	(b)
51.	(a)	**52.**	(d)	**53.**	(a)	**54.**	(c)	**55.**	(b)	**56.**	(c)	**57.**	(a)	**58.**	(d)	**59.**	(c)	**60.**	(c)
61.	(b)	**62.**	(b)	**63.**	(d)	**64.**	(b)	**65.**	(a)	**66.**	(c)	**67.**	(a)	**68.**	(b)	**69.**	(a)	**70.**	(b)
71.	(c)	**72.**	(d)	**73.**	(d)	**74.**	(d)	**75.**	(a)	**76.**	(d)	**77.**	(a)	**78.**	(d)	**79.**	(a)	**80.**	(c)

HINTS AND SOLUTIONS

1. Given that $z = (x + a)(y + b)$

Differentiate partially w.r.t. x, $p = \dfrac{\partial z}{\partial x} = y + p$

again differentiate partially w.r.t. y, $q = \dfrac{\partial z}{\partial y} = x + a$

Eliminate a and b from the above equations we get

$$z = pq$$

3. The partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^3 = 0 \text{ so its order is 2.}$$

8. Given that $z = f\left(\dfrac{y}{x}\right)$

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

and $$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\frac{1}{x}$$

Dividing these we get

$$\frac{p}{q} = -\frac{y}{x} \qquad \Rightarrow \qquad px + qy = 0$$

10. Given that $z = f(x^2 + y^2)$

$$p = \frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and $$q = \frac{\partial z}{\partial y} = 2yf'(x^2 + y^2)$$

Dividing we get $yp = xq$

13. Given differential equation is $zxp - zyq = y^2 - x^2$

So, Lagrange's auxiliary equation is

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$$

Taking first two we get $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$

Integrating $\log x + \log y = \log c_1$

$\Rightarrow \qquad xy = c_1$

Chosing x, y, z as multipliers we get

$\dfrac{xdx + ydy + zdz}{0}$ i.e. $xdx + ydy + zdz = 0$

Integrating we get

$$x^2 + y^2 + z^2 = c_2$$

So solution is $f(xy, x^2 + y^2 + z^2) = 0$

15. Given $p + q = \dfrac{z}{a}$

Lagranges auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{z}{a}}$$

By first and second $dx = dy \Rightarrow x - y = c_1$

By second and third $dy = \dfrac{adz}{z} = y = a\log z + \log c_2$

$$\log c_2 = y - a\log z = \log e^y - \log z^a = \log\frac{e^y}{z^a}$$

$\Rightarrow \qquad z = c_3 e^{y/a}$

So solution is $z = e^{y/a} f(x - y)$

17. Given $zy^2p - zx^2q = x^2y$

So Lagrange's auxiliary equation is

$$\frac{dx}{zy^2} = \frac{dy}{-zx^2} = \frac{dz}{x^2y}$$

By first and second $x^2dx + y^2dy = 0$

Integrating we get $x^3 + y^3 = c_1$

19. Lagrange's auxiliary equation is $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

By first and second $\log x = \log y + \log c_1$

$$\Rightarrow \qquad \frac{x}{y} = c_1$$

and by second and third $\log y = \log z + \log c_2$

$$\Rightarrow \qquad \frac{y}{z} = c_2$$

So solution is $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

20. Given $p^2 + q^2 = 1$

i.e. $\qquad f(p, q) = 0 \qquad$ so $\quad a^2 + b^2 = 1$

So its solution is $z = ax + by + c$

i.e. $\qquad z = ax + \sqrt{1 - a^2}y + c$

differentiate w.r.t. a and c we get

$0 = x$ and $0 = 1$ which is not possible so no singular solution exist.

21. Given $p^2 = zq$, put $p = \frac{dz}{dX}$ and $q = a\frac{dz}{dX}$

where $X = x + ay$

$$\left(\frac{dz}{dX}\right)^2 = az\frac{dz}{dX}$$

$$\Rightarrow \quad \frac{dz}{dX} = az \quad \Rightarrow \quad \frac{dz}{z} = aX$$

Integrating $\log z = aX + \log b$ or $z = be^{aX}$

i.e., $\quad z = be^{a(x+ay)} = be^{ax+a^2y}$

25. Given $-p_1 + p_2 + p_3 = 1$

Lagrange auxiliary equations are

$$\frac{dx_1}{-1} = \frac{dx_2}{1} = \frac{dx_3}{1} = \frac{dz}{z}$$

By (i) and (ii) $x_1 + x_2 = c_1$

By (i) and (iii) $x_1 + x_3 = c_2$

By (i) and(iv) $x_1 + z = c_3$

So solution is $f(x_1 + x_2, x_1 + x_3, x_1 + z) = 0$

26. The family of surfaces orthogonal to the family of surface $Pp + Qq = R$ is $Pdx = Qdy + Rdz = 0$

so we have $(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0$

Integrating we get

$$xy + yz + zx = c$$

28. Given $q = e^{-\frac{p}{a}}$ so solution is $z = ax + by + c$

where $d = e^{-\frac{b}{a}}$

So solution is $z = ax + ye^{-\frac{b}{a}} + c$

30. Given $p(1 + q) = qz$

put $\qquad p = \frac{dz}{dX}, q = \frac{adz}{dX}$

where $\qquad X = x + ay$

$$\frac{dz}{dX}\left(1 + a\frac{dz}{dX}\right) = a\left(\frac{dz}{dX}\right)z$$

$$\Rightarrow \qquad 1 + a\frac{dz}{dX} = az$$

$$\frac{adz}{az - 1} = dX \quad \Rightarrow \quad \log(az - 1) = X + \log b$$

$$\Rightarrow \quad az - 1 = be^X \Rightarrow \quad az = 1 + be^{x+ay}$$

36. Given equation is $z = px + qy + \log pq$

This is Clairant's equation so its solution is

$$z = ax + by + \log ab \qquad \text{...(1)}$$

Differentiating w.r.t. a and b respectively we get

$$0 = x + \frac{b}{ab} \quad \Rightarrow \quad a = \frac{-1}{x} \text{ and } b = \frac{-1}{y}$$

put in (1) we get singular solution is

$$z = -1 - 1 + \log\frac{1}{xy} = -2 - \log xy$$

38. Given that $z = px + qy - p^2q$

Its solution is $z = ax + by - a^2b \qquad$ ...(1)

Diff. partially w.r.t. a and b respectively we get

$$0 = x - 2ab \text{ and } 0 = y - a^2$$

i.e., $\qquad a = \sqrt{y}$ and $b = \frac{x}{2\sqrt{y}}$

put these values in equation (1) we get the singular solution is

$$z = x\sqrt{y} + \frac{1}{2}x\sqrt{y} - y\frac{x}{2\sqrt{y}} = x\sqrt{y}$$

40. Given that $z = Ae^{pt}\sin px$

then $\frac{\partial z}{\partial x} = Ape^{pt}\cos px$ and $\frac{\partial z}{\partial t} = Ape^{pt}\sin px$

$$\frac{\partial^2 z}{\partial x^2} = -Ap^2e^{pt}\sin px \text{ and } \frac{\partial^2 z}{\partial t^2} = Ap^2e^{pt}\sin px$$

so, $\qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

43. Given $z = f(x + ay) + \phi(x - ay)$

then $\frac{\partial z}{\partial x} = f'(x + ay) + \phi'(x - ay)$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$

Again $\dfrac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$

So we get $a^2\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

45. Given that $f = px + qy - pq$ so Charpit's equations

$$\frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}} = ...$$

or $\dfrac{dx}{-(x-q)} = \dfrac{dy}{-(y-p)} = ...$

or $\dfrac{dx}{q-x} = \dfrac{dy}{p-y}$

47. Let $f = p^2 + q^2 - 2px - 2qy - 1 = 0$

Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = ...$$

or $\dfrac{dp}{-2p + p.0} = \dfrac{dq}{-2q + q.0} = ...$

or $\dfrac{dp}{p} = \dfrac{dq}{q}$

50. Given that $p^2q^2(px + qy - z) = 2$

or $z = px + qy - \dfrac{2}{p^2q^2}$ which is a Clairauts equation

so its solution is

$$z = ax + by - \frac{2}{a^2b^2}$$

52. Given that $f = px + qy + z - xq^2$

Charpit's auxiliary equation is

$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}} = ...$$

or $\dfrac{dq}{q + q.1} = \dfrac{dx}{-x}$

or $\dfrac{dq}{2q} = -\dfrac{dx}{x} \Rightarrow \dfrac{dx}{x} = -\dfrac{dq}{2q}$

54. Given that $yp = 2xy + \log q$

or $p - 2x = \dfrac{1}{y}\log q = a$

$\Rightarrow$ $p = 2x + a$ and $q = e^{ay}$

so put these value in $dz = pdx + qdy$

i.e. $dz = (2x + a)\,dx + e^{ay}dy$

Integrating we get

$$z = x^2 + ax + \frac{e^{ay}}{a} + b$$

56. Given $pq = xy \Rightarrow \dfrac{p}{x} = \dfrac{y}{q} = a$

so $p = ax$ and $q = \dfrac{y}{a}$

put in $dz = pdx + qdy = axdx + \dfrac{y}{a}dy$

Integrating we get $2az = a^2x^2 + y^2 + 2ab$

58. Given $pe^y = qe^x \Rightarrow pe^{-x} = qe^{-y} = a$

so $p = ae^x$ and $q = ae^y$

put these values in $dz = pdx + qdy = ae^x dx + ae^y dy$

Integrating we get $z = ae^x + ae^y + b$

63. Given that $(x + 2z)\,p + (4zx - y)\,q = 2x^2 + y$

If we choose $y, x, -2z$ as multipliers then we get $\dfrac{ydx + xdy - 2zdz}{0}$

Since Lagrange's auxiliary equations

$$\frac{dx}{x + 2z} = \frac{dy}{4zx - y} = \frac{dz}{2x^2 + y}$$

i.e. $xdy + ydx - 2zdz = 0$

Integrating $xy - z^2 = c$

65. The Lagrange's auxiliary equations are

$$\frac{dx}{2y\,(z-3)} = \frac{dy}{2x - 2} = \frac{dz}{y\,(2x-3)}$$

If we choose $\dfrac{1}{2}, y, -1$ are multipliers we get

$$\frac{\frac{1}{2}dx + ydy - dz}{yz - 3y + 2xy - yz - 2xy + 3y}$$

So we get $\dfrac{1}{2}dx + ydy - dz = 0$

Integrating we get $x + y^2 - 2z = c$

67. Given that $f = z^2 - pqxy$

Charpit's auxiliary equation are

$$\frac{\partial x}{-\dfrac{\partial f}{\partial p}} = \frac{\partial y}{-\dfrac{\partial f}{\partial q}} = ...$$

so we get $\dfrac{dx}{qxy} = \dfrac{dy}{pxy} = ...$

or $\dfrac{dx}{q} = \dfrac{dy}{p}$

71. Given differential equation is $z = ax + a^2y^2 + b$

$$\frac{\partial z}{\partial x} = a = p \text{ and } \frac{\partial z}{\partial y} = 2a^2y = q$$

Eliminate a between $p = a$ and $q = 2a^2y$ we get

$$q = 2p^2y \text{ so its degree is 2.}$$

72. Given $zp + x = 0$ its Lagrange's equation is

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

By second term $dy = 0 \Rightarrow y = c_1$

and by first and third term $xdx + zdz = 0$

Integrating $x^2 + z^2 = c_2$

so solution is $x^2 + z^2 = f(y)$

74. Given $p + q = 1$

so its solution is $z = ax + by + c$

where $a + b = 1$

i.e. $b = 1 - a$

So required solution is $z = ax + (1 - a)\,y + c$

Integrating w.r.t. c we have $0 = 1$

so not S.S. exist.

77. $z = f(x - y)$

Differentiating w.r.t. x and y respectively

$p = f'(x - y)$ and $q = -f'(x - y)$

so, $\qquad p + q = 0$

❍❍❍

Unit-II

CHAPTER 9

Linear Partial Differential Equation of Second and Higher Order with Constant Coefficients

LINEAR PARTIAL DIFFERENTIAL EQUATION

1. A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients all being constants or functions of x and y is called linear partial differential equation.

   For example,

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots A_n \frac{\partial^n z}{\partial y^n} + \dots .$$
$$+M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + Tz = f(x, y) \; ...(1)$$

   Where the coefficients $A_1, A_2, \dots, A_n, M, N, T$ are constants or functions of x and y.

2. If the coefficients of various terms in equation (1) are constants then it is called a linear partial differential equation with constant coefficients.

3. A linear differential (partial) equation of order n with constant coefficients is

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

   where $A_1, A_2 \dots . A_n$ are all constants.

   or $(D^n + A_1 D^{n-1} D' + \dots . + A_n D'^n) z = f(x, y)$

   or $f(D, D')z = f(x, y)$

   Here $f(D, D')$ is a homogeneous function in D, D' of degree n.

4. If u is the complementary function and v a particular integral of a linear partial differential equation $f(D, D')z = f(x, y)$ then $u + v$ is a general solution of the equation.

DETERMINATION OF THE COMPLEMENTARY FUNCTION (C.F.)

1. The general solution of $f(D, D')z = 0$ which contains the correct number of arbitrary functions is called the complementary function of the partial differential equation.

2. consider $f(D, D')z = 0$

   $i.e., (D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2$

$$+ \dots + A_n D'^n)\, z = 0 \quad ...(1)$$

   Then auxiliary equation (A.E.) is

$$m^n + A_1 m^{n-1} + \dots + A_n = 0 \qquad ...(2)$$

   It contains n roots say $m_1, m_2, \dots m_n$

Case I : If all roots are distinct then general solution of (1) is

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$
$$+ \dots + \phi_n(y + m_n x)$$

where $\phi_1, \phi_2, \dots, \phi_n$ are arbitrary functions.

Case II : If a root m repeated r times then C.F. is

$$\phi_1(y + mx) + x\,\phi_2(y + mx) + \dots + x^{r-1}\phi_r(y + mx)$$

DETERMINATION OF PARTICULAR INTEGRAL (P.I.)

1. Consider $f(D, D')\, z = f(x, y)$ so its P.I. is

$$\frac{1}{f(D, D')} f(x, y)$$

   $\frac{1}{f(D, D')} V$ is defined as the function which given V when it is operated upon by $f(D, D')$.

2. The shortcut method for P.I. of $f(D, D')z = \phi(ax + by)$ is given by

   Case I : When $f(a, b) \neq 0$ and $f(D, D')$ is a homogeneous function of degree n,then

$$\text{P.I.} = \frac{1}{f(a, b)} \iiint \dots\dots \int f(u)\, du du \dots du$$

   Where $u = ax + by$ and R.H.S. contains a multiple integral of n^{th} order.

Case II : When $f(a, b) = 0$ we have

$$\frac{1}{(bD - aD')^n}\,\phi(ax + by) = \frac{x^n}{n!\,b^n}\,\phi(ax + by)$$

3. The general method of finding the P.I. of

$$(D - mD')\,z = \phi(x, y)$$

or $\quad p - mq = \phi(x, y)$

Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x, y)}$$

By first two we get $y + mx = a$ (constant)

and by first and last term we get

$$dz = \phi(x, y)\,dx = \phi(x, a - mx)\,dx$$

$$z = \int \phi(x, a - mx)\,dx$$

Thus $z = \dfrac{1}{(D - mD')}\,\phi(x, y) = \int \phi(x, a - mx)\,dx$

NON-HOMOGENEOUS LINEAR EQUATION WITH CONSTANT COEFFICIENT

1. A linear differential equation which is not homogeneous is called a non-homogeneous linear equation.

Consider $f(D, D')z = f(x, y)$ ...(1)

Where $f(D, D')$ is non-homogeneous.

Let

$$f(D, D') = (D - m_1 D' - k_1)\,(D - m_2 D' - k_2)\,\ldots(D - m_n D' - k_n)$$

Then C.F. of equation (1) is

$$z = e^{k_1 x}\phi_1(y + m_1 x) + e^{k_2 x}\phi_2(y + m_2 x) + \ldots + e^{k_n x}\phi_n(y + m_n x)$$

2. If $(D - mD' - k)$ occurs r times in $f(D, D')$ then the C.F. corresponding to this factor is

$$e^{kx}[\phi_1(y + mx) + x\phi_2(y + mx) + \ldots + x^{r-1}\phi_r(y + mx)]$$

If $f(D, D')$ is irreducible then a trial method is used to find solutions.

3. To find the particular integral P.I.

so $\quad f(D, D')z = f(x, y)$ we have

P.I. is $\dfrac{1}{f(D, D')}\,f(x, y)$

Case I : $\dfrac{1}{f(D, D')}\,e^{ax+by} = \dfrac{1}{f(a, b)}\,e^{ax+by}$ if $f(a, b) \neq 0$

Case II : $\dfrac{1}{f(D, D')}\sin(ax + by)$, we put

$$D^2 = -a^2,\ DD' = -ab,\ D'^2 = -b^2$$

provided eliminator is not zero. Similarly for $\cos(ax + by)$

Case III : $\dfrac{1}{f(D, D')}\,x^m y^n = [f(D, D')]^{-1} x^m y^n$

Which can be evaluated after expanding $[f(D, D')]^{-1}$ in ascending power of D or D'.

Case IV : $\dfrac{1}{f(D, D')}\,(e^{ax+by}.V)$

$$= e^{ax+by}\,\frac{1}{f(D + a, D' + b)}\,V$$

Case V : If $f(a, b) = 0$ then

P.I. $\dfrac{1}{f(D, D')}\,f(ax + by) = \dfrac{x}{(f')_{(a,\,b)}}\,f(ax + by)$

Where $\quad f' = \dfrac{\partial f}{\partial D}$

If $\quad (f')_{(a,\,b)} = 0$ then

P.I. is $\dfrac{x^2}{(f'')_{(a,\,b)}}\,f(ax + by)$ and so on.

EQUATIONS REDUCIBLE TO LINEAR FORM WITH CONSTANT COEFFICIENTS

1. Consider $A_0 x^n \dfrac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y\,\dfrac{\partial^n z}{\partial x^{n-1}\partial y}$

$$+ \ldots A_n y_n \frac{\partial^n z}{\partial y_n} = f(x, y)$$

Put $x = e^u$ and $y = e^v$

i.e. $u = \log x$ and $v = \log y$ we get

$$x\,\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} \equiv D$$

$$x^2\,\frac{\partial^2}{\partial x^2} \equiv D(D - 1)$$

$$x^3\,\frac{\partial^3}{\partial x^3} \equiv D\,(D - 1)\,(D - 2)$$

Similarly $y = \dfrac{\partial}{\partial y} \equiv D'z,\ D' \equiv \dfrac{\partial}{\partial v}$

$$y^2\,\frac{\partial^2}{\partial y^2} \equiv D'\,(D' - 1),\ y^3\,\frac{\partial^3}{\partial y^3} \equiv D'\,(D' - 1)\,(D' - 2)$$

$$xy\frac{\partial^2}{\partial x\partial y} \equiv DD'$$

Substituting in given equation it reduces to

$$f(D, D')\,z = V$$

Which is an equation having constant coefficients.

C.F. OF NON-HOMOGENEOUS LINEAR PDE WHEN LINEAR FACTORS ARE NOT POSSIBLE

1. Consider $(D^2 - D')z = 0$ ...(1)

Let its trial solution is $z = Ae^{hx+ky}$ ...(2)

then $D^2z = Ah^2e^{hx+ky}$ and $D'z = kAe^{hx+ky}$

put these values in (1) we get

$$(h^2 - k)\,Ae^{hx+ky} = 0$$

$$\Rightarrow \qquad h^2 = k \qquad \text{...(3)}$$

So, C.F. of (1) is

$$z = Ae^{hx+h^2y}$$

Since all values of k satisfy equation (1) so general solution of (1) is

$$z = \Sigma Ae^{hx+h^2y}$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The general solution of the differential equation $(D^3 - 4D^2D' + 3DD'^2)\,z = 0$ is :

(a) $c_1 + c_2e^x + c_3e^{3x}$

(b) $\phi_1(y) + \phi_2(y + x) + \phi_3(y + 3x)$

(c) $\phi_1(x) + \phi_2(y + x) + \phi_3(y + 3x)$

(d) $(c_1 + c_2x)\,e^x + c_3e^{3x}$

2. The complementary function of $\log s = x$ is :

(a) $\phi_1(y) + y\phi_2(y)$ (b) $\phi_1(x) + x\phi_2(x)$

(c) $\phi_1(x) + \phi_2(y)$ (d) $\phi_1(x) + y\phi_2(x)$

3. The auxiliary equation of $2r + 5s + 2t = 0$ is :

(a) $2m^2 + 5m + 2 = 0$ (b) $2m^2 + 5m - 2 = 0$

(c) $2m^2 - 5m + 2 = 0$ (d) $2m^2 - 5m - 2 = 0$

4. The roots of A.E. of $(D^4 - D'^4)\,z = 0$ are :

(a) $1, 1, 1 + i, 1 - i$ (b) $1, -1, 1 + i, 1 - i$

(c) $1, 1, \pm i$ (d) $1, -1, \pm i$

5. The solution of the differential equation $r^2 - s^2 = 0$ is :

(a) $\phi_1(y + ix) + \phi_2(y - ix)$ (b) $c_1e^x + c_2e^{-x}$

(c) $(c_1 + c_2x)\,e^{-x}$ (d) $\phi_1(y + x) + \phi_2(y - x)$

6. The particular integral of $(D^2 - a^2D'^2)\,z = x^2$ is :

(a) $\frac{x^2}{12}$ (b) $\frac{x^3}{6}$

(c) $\frac{x^3}{12}$ (d) $\frac{x^4}{12}$

7. The P.I. of $(D^2 - DD' - 6D'^2)\,z = xy$ is :

(a) $\frac{xy}{2} + \frac{x^2}{4}$ (b) $\frac{x^2y + x^3}{6}$

(c) $\frac{x^3y}{6} + \frac{x^4}{24}$ (d) $\frac{x^2y}{6} + \frac{x^4}{12}$

8. $\frac{1}{D_1^3 + D_2^3 + D_3^3 - 3D_1D_2D_3}x^3$ is equal to :

(a) $\frac{x^6}{12}$ (b) $\frac{x^6}{120}$ (c) $\frac{x^6}{60}$ (d) $\frac{x^6}{80}$

9. The solution of $(D^2 - 4DD' + 4D'^2)\,z = 0$ is :

(a) $(c_1 + c_2x)\,e^{2x}$ (b) $c_1e^x + c_2e^{2x}$

(c) $\phi_1(y + 2x) + \phi_2(y + x)$

(d) $\phi_1(y + 2x) + x\phi_2(y + 2x)$

10. The solution of $(D^3 - 3D^2D' + 2DD'^2)\,z = 0$ is :

(a) $\phi_1(y) + \phi_2(x) + \phi_3(y + x)$

(b) $\phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$

(c) $\phi_1(y) + \phi_2(x) + \phi_3(y + 2x)$

(d) $\phi_1(x) + \phi_2(y + x) + \phi_3(y + 2x)$

11. The roots of A.E. of $(D^4 - 2D^3D' + 2DD'^3 - D'^4)\,z = 0$ are :

(a) $1, 1, 1, -1$ (b) $1, 1, 2, -1$

(c) $1, 2, -1, -2$ (d) $1, 1, 1, 1$

12. The P.I. of the differential equation $r - t = x - y$ is :

(a) $\frac{x^3}{6} - \frac{x^2y}{2}$ (b) $\frac{x^2}{6} - \frac{y}{2}$

(c) $\frac{y^3}{6} - \frac{xy^2}{2}$ (d) $\frac{x^3}{6} + \frac{xy}{2}$

13. The P.I. of $4r - 4s + t = 16\log(x + 2y)$ is :
(a) $x^2 \log(x + 2y)$ (b) $2x \log(x + 2y)$
(c) $x \log(x + 2y)$ (d) $2x^2 \log(x + 2y)$

14. The value of $\frac{1}{2D^2 - DD' - 3D'^2} e^{x-y}$ is :
(a) $\frac{e^{x-y}}{5}$ (b) $\frac{xe^{x-y}}{5}$ (c) $\frac{x^2e^{x-y}}{5}$ (d) $\frac{xe^{y-x}}{5}$

15. The particular integral of $r - 2as + a^2t = f(y + ax)$ is :
(a) $f(y + ax)$ (b) $\frac{f(y + ax)}{2}$
(c) $\frac{xf(y + ax)}{2}$ (d) $\frac{x^2f(y + ax)}{2}$

16. The P.I. of $r + s - 2t = (2x + y)^{1/2}$ is :
(a) $\frac{(2x + y)^{1/2}}{15}$ (b) $\frac{(2x + y)^{3/2}}{15}$
(c) $\frac{(2x + y)^{5/2}}{15}$ (d) $\frac{(2x + y)^{7/2}}{15}$

17. The roots of A.E. of $(D^3 - 7DD'^2 - 6D'^3)\, z = 0$ are :
(a) 1, 2, 3 (b) $-1, -2, -3$
(c) $-1, 2, -3$ (d) $-1, -2, 3$

18. P.I. of $r + s - 6t = y\cos x$ is :
(a) $\cos y - x\sin y$ (b) $\cos y + y\sin x$
(c) $\sin x - y\cos x$ (d) $\sin x + x\cos x$

19. $\frac{1}{D^2 - DD' - 2D'^2}(y - 1)\, e^x$ is equal to :
(a) e^x (b) xe^x
(c) ye^x (d) $(x + y)\, e^x$

20. The solution of $p + q = \sin x$ is :
(a) $\phi(y - x) - \cos x$ (b) $\phi(y - x) + \sin x$
(c) $\phi(y - x) + \cos x$ (d) $\phi(y + x) - \sin x$

21. The solution of $(D - mD' - k)\, z = 0$ is :
(a) $e^{-kx}\phi(y + mx)$ (b) $e^{-kx}\phi(y - mx)$
(c) $e^{kx}\phi(y - mx)$ (d) $e^{kx}\phi(y + mx)$

22. The solution of $(D - D'^2)\, z = 0$ is :
(a) $\phi_1(y + x) + \phi_2(x)$ (b) $\Sigma Ae^{k^2x + ky}$
(c) $\phi(y + x) + \phi_2(y)$ (d) $\Sigma Ae^{kx + ky}$

23. The value of $\frac{1}{(D - D' - 1)(D - D' - 2)} x$ is :
(a) $\frac{x + 3}{2}$ (b) $\frac{2x + 3}{4}$
(c) $\frac{2x - 3}{4}$ (d) $\frac{x - 3}{2}$

24. The solution of $s + p - q = 0$ is :
(a) $e^{-x}\phi_1(y) + e^y\phi_2(x)$ (b) $e^x\phi_1(x) + e^{-y}\phi_2(y)$
(c) $e^x\phi_1(y) + e^{-y}\phi_2(x)$ (d) $e^{-x}\phi_1(x) + e^y\phi_2(y)$

25. The P.I. of $(D^2 - D')\, z = 2y - x^2$ is :
(a) xy (b) x^2y (c) xy^2 (d) $x + y$

26. $\frac{1}{D + D'}\sin x$ is equal to :
(a) $-\sin x$ (b) $\cos x$
(c) $-\cos x$ (d) $\sin x$

27. The P.I. of $(D_1^3 + D_2^3 + D_3^3 - 3D_1D_2D_3)\, z = -3xyz$ is :
(a) $\frac{xyz}{8}$ (b) $\frac{xy^2z}{8}$ (c) $\frac{x^2yz}{8}$ (d) $\frac{x^2y^2z^2}{8}$

28. The value of $\frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$ is :
(a) $\frac{(x + y)^2}{18}$ (b) $\frac{(x + y)^3}{36}$
(c) $\frac{(x + y)^2}{36}$ (d) None of these

29. The value of $\frac{1}{D^2 - 2DD' - 15D'^2} 12xy$ is :
(a) $x^4 + 2x^3y$ (b) $x^3 + 2x^4y$
(c) $x^4 + 2xy^3$ (d) None of these

30. The value of $\frac{1}{f(D, D')} e^{ax + by}$, provided $f(a, b) \neq 0$ is :
(a) $\frac{1}{f(-a, -b)} e^{ax + by}$ (b) $\frac{1}{f(a,b)} e^{a + b}$
(c) $\frac{1}{f(a, b)} e^{a^2 + b^2}$ (d) $\frac{1}{f(a, b)} e^{ax + by}$

31. The P.I. of $(D^2 - D'^2 - 3D + 3D')\, z = e^{x + 2y}$ is :
(a) $e^{x + 2y}$ (b) $xe^{x + 2y}$
(c) $-xe^{x + 2y}$ (d) None of these

32. The auxiliary equation of $x^2r + 2xys + y^2t = 0$ is :
(a) $(D - D')(D - D' - 1)\, z = 0$
(b) $(D - D')(D - D' + 1)\, z = 0$
(c) $(D + D')(D - D' + 1)\, z = 0$
(d) $(D + D')(D + D' - 1)\, z = 0$

33. The P.I. of $x^2r - y^2t = xy$ is :
(a) xy (b) $\frac{x}{y}\log x$
(c) xye^x (d) $xy\log x$

34. The P.I. of $x^2r - y^2t + px - qy = \log x$ is :

(a) $\dfrac{e^{3x}}{6}$ (b) $\dfrac{(\log x)^3}{6}$

(c) $\dfrac{(\log x)^2}{6}$ (d) $\dfrac{e^{2x}}{6}$

35. The solution of $x^2r - y^2t = 0$ is :

(a) $\phi_1(xy) + x\phi_2(xy)$ (b) $\phi_1(xy) + \phi_2\left(\dfrac{y}{x}\right)$

(c) $\phi\,(xy) + x\phi_2\left(\dfrac{y}{x}\right)$ (d) $\phi_1(xy) + y\phi_2\left(\dfrac{y}{x}\right)$

36. If $z = \Sigma A e^{hx+ky}$ be the solution of $(D^2 - D')\,z = 0$ then :

(a) $h^2 = k$ (b) $h = k^2$

(c) $h^3 = k$ (d) $h = k$

37. The equation $x^2\dfrac{\partial^2 z}{\partial x^2} + 4xy\dfrac{\partial^2 z}{\partial y^2} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$ can be reduced into linear equation with constant coefficients if we substitute :

(a) $x = X + h,\ y = Y + k$ (b) $x = e^X,\ y = e^Y$

(c) $x = \log X,\ y = \log Y$ (d) $x = e^{2X},\ y = e^{2Y}$

38. The solution of $r - a^2t = 0$ is :

(a) $c_1e^{ax} + c_2e^{-ax}$

(b) $\phi_1(y + ax) + x\phi_2(y + ax)$

(c) $\phi_1(y + ax) + \phi_2(y - ax)$

(d) $(c_1 + c_2x)\,e^{-ax}$

39. The roots a A.E. of $r + (a + b)\,s + abt = xy$ is :

(a) a, b (b) $a, -b$

(c) $-a, b$ (d) $-a, -b$

40. The value of $\dfrac{1}{D^2 + D'^2}\{-4\pi\,(x^2 + y^2)\}$ is :

(a) $-2x^2y^2$ (b) $-2\pi^2x^2y^2$

(c) $-2\pi x^2y^2$ (d) $-2\pi xy$

41. $\dfrac{1}{(bD - aD')}x^r\phi\,(ax + by)$ is equal to :

(a) $\dfrac{x^r}{r+1}\phi(ax + by)$ (b) $\dfrac{x^{r+1}}{r+1}\phi(ax + by)$

(c) $\dfrac{x^{r+1}}{b\,(r+1)}\phi\,(ax + by)$ (d) $\dfrac{x^r}{b\,(r+1)}\phi\,(ax + by)$

42. The solution of $r + t = 0$ is :

(a) $\phi_1(y + ix) + \phi_2(y - ix)$

(b) $\phi_1(y + x) + \phi_2(y - x)$

(c) $\phi_1(y - x) + \phi_2(y - x)$

(d) $\phi_1(y + ix) + x\phi_2(y - ix)$

43. The value of $\dfrac{1}{(D^2 - 5DD' + 4D'^2)}\sin\,(4x + y)$ is :

(a) $\dfrac{y}{2}\cos\,(4x + y)$ (b) $-\dfrac{y_1}{2}\cos(4x + 5)$

(c) $-\dfrac{x}{3}\cos\,(4x + y)$ (d) $-\dfrac{y}{3}\cos\,(4x + y)$

44. $\dfrac{1}{D - mD'}\phi\,(x, y)$ is equal to :

(a) $\int \phi(x,\ a - mx)\ dx$ (b) $\int \phi(x,\ a + mx)\ dx$

(c) $\int \phi(x,\ a - my)\ dx$ (d) $\int \phi\,(x,\ a + my)\ dx$

45. The solution of $(4D^2 + 12DD' + 9D'^2)\ z = 0$ is :

(a) $\phi_1(2y + 3x) + \phi_2(2y - 3x)$

(b) $\phi_1(2y - 3x) + x\phi_2(2y - 3x)$

(c) $e^{3x}\phi_1(2y) + e^{-3x}\phi_2(2y)$

(d) $[\phi_1(2y) + x\phi_2(2y)]\ e^{-3x}$

46. The P.I. of $r + 3s + 2t = 6\,(x + y)$ is :

(a) $3xy - 2x^2$ (b) $xy - 3x^2$

(c) $\dfrac{(x - y)^3}{6}$ (d) $3x^2y - 2x^3$

47. The value of $\dfrac{1}{(bD - aD')^2}\phi(ax + by)$ is :

(a) $\dfrac{x^2}{2!\,b^2}\phi\,(ax + by)$ (b) $\dfrac{x}{b}\phi\,(ax + by)$

(c) $\dfrac{x^2}{2!\ b^2}\phi'(ax + by)$ (d) $\dfrac{x^2}{2!\ b^2}\phi''(ax + by)$

48. $\dfrac{1}{(D - D')^2}\phi\,(x + y)$ is equal to :

(a) $\dfrac{x}{2}\phi\,(x + y)$ (b) $\dfrac{x^2}{2}\phi\,(x + y)$

(c) $\dfrac{x^3}{3}\phi\,(x + y)$ (d) $\dfrac{x}{3}\phi\,(x + y)$

49. The P.I. of $r - 2s + t = \sin\,(2x + 3y)$ is :

(a) $2\sin\,(2x + 3y)$ (b) $\dfrac{1}{2}\sin\,(2x + 3y)$

(c) $\dfrac{1}{3}\sin\,(2x + 3y)$ (d) $-\sin\,(2x + 3y)$

50. The P.I. of $r + s - 6t = y\sin x$ is :

(a) $-y\sin x + \cos x$ (b) $-x\sin y - \cos y$

(c) $-y\sin x - \cos x$ (d) $-x\sin y + \cos y$

51. $\frac{1}{f(D, D')}(e^{ax+by}.V)$ is equal to :

(a) $e^{ax+by}\frac{1}{f(D, D)}V$ (b) $e^{ax+by}\frac{1}{f(a, b)}V$

(c) $Ve^{ax+by}\frac{1}{f(D, D')}1$

(d) $e^{ax+by}\frac{1}{f(D+a, D'+b)}V$

52. If $z = Ae^{hx+ky}$ be the solution of $(2D^4 - 3D^2D' + D'^2)\, z = 0$ then :

(a) $h = k^2$ (b) $k = h^2$

(c) $h = -k$ (d) $h = k$

53. $\frac{1}{(D^2 - DD' + D' - 1)}\cos(x + 2y)$ is equal to :

(a) $\frac{1}{2}\cos(x + 2y)$ (b) $-\frac{1}{2}\sin(x + 2y)$

(c) $\frac{1}{2}\sin(x + 2y)$ (d) $-\frac{1}{2}\cos(x + 2y)$

54. The complementry function of $(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = xy$ is :

(a) $\phi(y + x) + e^{2x}\phi_2(y + 2x)$

(b) $\phi_1(y - x) + e^{2x}\phi_2(y + 2x)$

(c) $\phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$

(d) $\phi_1(y + x) + e^{-2x}\phi_2(y + 2x)$

55. The value of $\frac{1}{D^2 - D'^2 + D + 3D' - 2}e^{x-y}$ is:

(a) $\frac{1}{4}e^{x-y}$ (b) $-\frac{1}{4}e^{x-y}$

(c) $-\frac{1}{6}e^{x-y}$ (d) $\frac{1}{6}e^{x-y}$

56. If ΣAe^{hx+ky} be the solution of $(D^2 - DD' - 2D)\, z = 0$, then k :

(a) h (b) $h - 1$ (c) $h - 2$ (d) $h - 3$

57. The roots of A.E. of $r + s - 6t = 0$ are :

(a) $-2, -3$ (b) $2, 3$ (c) $-2, 3$ (d) $2, -3$

58. The solution of $r - s + q - z = 0$ is :

(a) $\phi_1(y) + e^{-x}\phi_2(y + x)$ (b) $e^x\phi_1(x) + e^{-x}\phi_2(y - x)$

(c) $e^x\phi_1(y) + e^x\phi_2(y + x)$ (d) $e^x\phi_1(y) + e^{-x}\phi_2(y + x)$

59. The P.I. of $(DD' + aD + bD' + ab)z = e^{mx+ny}$ is :

(a) $\frac{e^{mx+ny}}{mn + ab}$ (b) $\frac{e^{mx+ny}}{(m + n)(a + b)}$

(c) $\frac{e^{mx+ny}}{(n + a)(m + b)}$ (d) $\frac{e^{mx+ny}}{(m - n)(a - b)}$

60. If ΣAe^{hx+ky} be the solution of $(D^2 - D' - 1)\, z = 0$ then :

(a) $k = h + 1$ (b) $k = h^2 - 1$

(c) $k = h - 1$ (d) $k = h^2 + 1$

61. $\frac{1}{(D^2 - DD' + D' - 1)}e^y$ is equal to :

(a) $-xe^y$ (b) xe^y

(c) $-ye^x$ (d) None of these

62. $\frac{1}{(D^2 - DD' + D' - 1)}e^{y-x}$ is equal to :

(a) $\frac{e^{y-x}}{2}$ (b) $\frac{-ye^{x-y}}{2}$

(c) $\frac{-x^2}{2}e^{y-x}$ (d) $-\frac{y^2}{2}e^{-x+y}$

63. $\frac{1}{(D^2 + DD' - 2D'^2 + 2D + 2D')}e^{2x+3y}$ is equal to :

(a) $-\frac{1}{8}e^{2x+3y}$ (b) $-\frac{1}{6}e^{2x+3y}$

(c) $\frac{1}{10}e^{2x+3y}$ (d) $-\frac{1}{10}e^{2x+3y}$

64. The P.I. of $r - s + q - z = \sin(x + 2y)$ is :

(a) $-\frac{1}{2}\cos(x + 2y)$ (b) $-\frac{1}{2}\sin(x + 2y)$

(c) $\frac{1}{2}\cos(x + 2y)$ (d) None of these

65. $\frac{1}{(D^2 - D' - 1)}x^2y$ is equal to :

(a) $x^2 - y^2 - 2y + 4$ (b) $x^2 + y^2 - 2x + y + 4$

(c) $x^2 + x^2y + 2y + 4$ (d) $x^2 - x^2y - 2y + 4$

66. If ΣAe^{hx+ky} be the solution of $r - s + 2q - z$ then :

(a) $k = \frac{1 - h^2}{2 - h}$ (b) $x = \frac{1 + h}{2 - h}$

(c) $k = \frac{1 - h}{2 - h}$ (d) $k = \frac{1 + h^2}{2 - h}$

67. The general solution of $(D - 2D'^2 - 1)\, z = 0$ is :

(a) $\Sigma Ae^{(2k+1)\, x+ky}$ (b) $\Sigma Ae^{(2k^2-1)\, x+ky}$

(c) $\Sigma Ae^{(2k-1)\, x+ky}$ (d) $\Sigma Ae^{(2k^2+1)\, x+ky}$

68. $\frac{1}{(D^2 - D')}(2y - x)$ is equal to :

(a) $xy - y$ (b) $xy^2 - y$

(c) $yx - y^2$ (d) $\frac{y^2}{x} - x$

69. If ΣAe^{hx+ky} be the solution of $(D^2-4DD'+D-1)\,z=0$, then $k=$:

(a) $\frac{h^2+2h-1}{4}$ (b) $\frac{h^2+h-1}{4h}$

(c) $\frac{h^2-4h+1}{2h}$ (d) None of these

70. The P.I. of $x^2r-y^2t+px-qy=(\log x)^2$ is :

(a) $\frac{\log x}{6}$ (b) $\frac{(\log x)^3}{6}$

(c) $\frac{(\log x)^4}{6}$ (d) $\frac{(\log x)^4}{12}$

71. $\frac{1}{(D-mD')}f(x,y)=$

(a) $\int f(x, c-mx)$ (b) $\int f(c, c-mx)$

(c) $\int f(x, c+mx)$ (d) $\int f(x, mx)$

72. The P.I. of $\log s = x+y$ is :

(a) $\log(x+y)$ (b) e^{x+y}

(c) $x+y$ (d) $\frac{(x+y)^2}{2}$

73. $\frac{1}{D^3-D'^3}x^3y^3$ is equal to :

(a) $\frac{x^3y}{9}+\frac{x^9}{10080}$ (b) $\frac{x^6y^3}{120}+\frac{x^6}{10080}$

(c) $\frac{x^3y+x^9}{10080}$ (d) $\frac{x^6y^3}{120}+\frac{x^9}{10080}$

74. The P.I. of $(x^2D^2-y^2D'^2)\,z=x^2y$ is :

(a) $\frac{1}{2}xy^2$ (b) xy^2 (c) yx^2 (d) $\frac{1}{2}x^2y$

75. $\frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}(x+y)^{1/2}$ is equal to :

(a) $-\frac{x}{3}(x+y)^{3/2}$ (b) $-\frac{x^2}{3}(x+y)^{3/2}$

(c) $-\frac{1}{3}(x+y)^{3/2}$ (d) $-y^2(x+y)^{3/2}$

76. The P.I. of $(D^2+D'^2)\,z=30\,(2x+y)$ is :

(a) $(2x+y)^3$ (b) $\frac{(2x+y)^3}{3}$

(c) $\frac{(2x+y)^3}{6}$ (d) None of these

77. $\frac{1}{D^2+D'^2}\cos mx \sin y$ is :

(a) $\frac{\cos(mx+ny)+\cos(mx-ny)}{(m+n)}$

(b) $\frac{\cos(mx+ny)+\cos(mx-ny)}{(m^2+n^2)}$

(c) $\frac{\cos(mx+ny)+\cos(mx-ny)}{2(m^2+n^2)}$

(d) None of these

78. The P.I. of $(D^2-2DD'+D'^2)\,z=x^3$ is :

(a) $\frac{x^5}{10}$ (b) $\frac{x^5}{20}$ (c) $\frac{x^5}{5}$ (d) $\frac{x^5}{15}$

79. $\frac{1}{(D^2-4DD'+3D'^2)}(x+3y)^{1/2}$ is :

(a) $\frac{(x+3y)^{1/2}}{60}$ (b) $\frac{(x+3y)^{3/2}}{20}$

(c) $\frac{(x+3y)^{5/2}}{60}$ (d) $\frac{(x+3y)^{7/2}}{120}$

80. $\frac{1}{(D^2+6DD'+D'^2)}e^x$ is :

(a) e^x (b) $\frac{xe^x}{y}$

(c) $(x+y)\,e^x$ (d) ye^x

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(a)	**4.**	(d)	**5.**	(d)	**6.**	(d)	**7.**	(c)	**8.**	(b)	**9.**	(d)	**10.**	(b)
11.	(a)	**12.**	(a)	**13.**	(d)	**14.**	(b)	**15.**	(d)	**16.**	(c)	**17.**	(d)	**18.**	(c)	**19.**	(c)	**20.**	(a)
21.	(d)	**22.**	(b)	**23.**	(b)	**24.**	(c)	**25.**	(b)	**26.**	(c)	**27.**	(d)	**28.**	(b)	**29.**	(a)	**30.**	(d)
31.	(c)	**32.**	(d)	**33.**	(d)	**34.**	(b)	**35.**	(c)	**36.**	(a)	**37.**	(b)	**38.**	(c)	**39.**	(d)	**40.**	(c)
41.	(c)	**42.**	(a)	**43.**	(c)	**44.**	(a)	**45.**	(b)	**46.**	(d)	**47.**	(a)	**48.**	(b)	**49.**	(d)	**50.**	(c)
51.	(d)	**52.**	(b)	**53.**	(c)	**54.**	(c)	**55.**	(b)	**56.**	(c)	**57.**	(b)	**58.**	(d)	**59.**	(c)	**60.**	(b)
61.	(a)	**62.**	(a)	**63.**	(d)	**64.**	(c)	**65.**	(d)	**66.**	(a)	**67.**	(d)	**68.**	(c)	**69.**	(b)	**70.**	(d)
71.	(a)	**72.**	(b)	**73.**	(d)	**74.**	(d)	**75.**	(b)	**76.**	(a)	**77.**	(c)	**78.**	(b)	**79.**	(c)	**80.**	(a)

HINTS AND SOLUTIONS

1. Given differential equation is

$$(D^3 - 4D^2D' + 3DD'^2)\, z = 0$$

or $(D^3 - 3D^2D' - D^2D' + 3DD'^2)\, z = 0$

or $[D^2(D - 3D') - DD'(D - 3D')]\, z = 0$

or $D\,(D - D')\,(D - 3D')\, z = 0$

so its solution is $\phi_1(y) + \phi_2(y + x) + \phi_3(y + 3x)$

2. Given $\log s = x \quad \Rightarrow \quad s = \dfrac{\partial^2 z}{\partial x \partial y} = x$

i.e. $DD' = x$

So C.F. is $\phi_1(x) + \phi_2(y)$

4. Given $(D^4 - D'^4)\, z = 0$

A.E. is $m^4 - 1 = 0$

$\Rightarrow \quad (m^2 + 1)\,(m^2 - 1) = 0$

$i.e.\, m = 1, -1, i, -i$

6. $$\frac{1}{(D^2 - a^2D'^2)}\, x^2 = \frac{1}{D^2}\left(1 - \frac{a^2D'^2}{D^2}\right)^{-1} x^2$$

$$= +\frac{1}{D^2}\left(1 + \frac{a^2D'^2}{D^2} + \ldots\right) x^2$$

$$= \frac{1}{D^2}x^2 = \frac{x^4}{12}$$

8. P.I. is $\dfrac{1}{D_1^3 + D_2^3 + D_3^3 - 3D_1D_2D_3}x^3$

$$= \frac{1}{D_1^3}\left(1 + \frac{D_2^3}{D_1^3} + \frac{D_3^3}{D_1^3} - \frac{3D_2D_3}{D_1^2}\right)^{-1} x^3$$

$$= \frac{1}{D_1^3}\left(1 - \frac{D_2^3}{D_1^3} - \frac{D_3^3}{D_1^3} + \frac{3D_2D_3}{D_1^2} + \ldots.\right) x^3 = \frac{1}{D_1^3}x^3$$

$$= \frac{x^6}{120}$$

10. Given differential equation is

$(D^3 - 3D^2D' + 2DD'^2)z = 0$

$D\,(D^2 - 3DD' + 2D'^2)z = 0$

or $D\,(D - D')\,(D - 2D')\, z = 0$

so its solution is

$$z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$$

12. Given that $r - t = \dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = x - y$

so its P.I. is $\dfrac{1}{D^2 - D'^2}(x - y)$

$$= \frac{1}{D^2}\left(1 - \frac{D'^2}{D^2}\right)^{-1} (x - y)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2} + \ldots\right)(x - y)$$

$$= \frac{1}{D^2}(x - y) = \frac{x^3}{6} - \frac{x^2y}{2}$$

14. $$\frac{1}{2D^2 - DD' - 3D'^2}e^{x-y}$$

$$= e^{x-y}\,\frac{1}{2\,(D + 1)^2 - (D + 1)\,(D' - 1) - 3\,(D' - 1)^2}.1$$

$$= e^{x-y}\,\frac{1}{2D^2 - 3D'^2 - DD' + 5D + 5D'}.1$$

$$= e^{x-y}\,\frac{1}{5D}\left(1 - \frac{3D'^2}{5D} - \frac{D'}{5} + \frac{2D}{5} + \frac{D'}{D}\right)^{-1}.1$$

$$= e^{x-y}\,\frac{1}{5D}.1 = \frac{x}{5}e^{x-y}$$

16. $$\frac{1}{D^2 + DD' - 2D'^2}(2x + y)^{1/2}$$

$$= \frac{1}{4 + 2 - 2}\iint u^{1/2}du\, du \text{ where } u = 2x + y$$

$$= \frac{1}{4}u^{5/2}\,\frac{2.2}{3.5} = \frac{u^{5/2}}{15} = \frac{(2x + y)^{5/2}}{15}$$

18. P.I. is $\dfrac{1}{D^2 + DD' - 6D'^2}y\cos x$

$$= \frac{1}{(D - DD')\,(D + 3D')}y\cos x$$

$$= \frac{1}{D - 2D'}\int (a + 3x)\, \cos x dx \text{ where, } y - 3x = a$$

$$-\frac{1}{D - 2D'}[a\sin x + 3x\sin x + 3\cos x]$$

$$= \frac{1}{D - 2D'}[(y - 3x)\, \sin x + 3x\sin x + 3\cos x]$$

$$= \int [(b - 2x)\sin x + 3\cos x]\; dx \text{ where, } y + 2x = b$$

$$= -b\cos x - 2\,(-x\cos x + \sin x) + 3\sin x$$

$$= -y\cos x + \sin x$$

20. $(D + D')z = \sin x$

C.F. is $\phi\,(y - x)$ and P.I. $\dfrac{1}{D + D'}\sin x = -\cos x$

So solution is $z = \phi\,(y - x) - \cos x$

22. Differential equation is $(D - D'^2)\ z = 0$ ...(1)

It has no factor so solution is $z = \Sigma A e^{hx+ky}$

put in (1) we get $(h - Ak^2)\ \Sigma A e^{hx+ky} = 0$

i.e. $h = k^2$

So required solution is $\Sigma A e^{k^2 x+ky}$

23. $$\frac{1}{(D-D'-1)\,(D-D'-2)}x$$

$$= \frac{1}{2}(1 - D + D')^{-1}\left(1 - \frac{D}{2} + \frac{D'}{2}\right)^{-1}.x$$

$$= \frac{1}{2}(1 + D - D' + \ldots)\left(1 + \frac{D}{2} - \frac{D'}{2} + \ldots.\right)x$$

$$= \frac{1}{2}\left(1 + \frac{3}{2}D + \ldots.\right)x = \frac{1}{2}\left(x + \frac{3}{2}\right) = \frac{2x+3}{4}$$

25. $$\frac{1}{D^2 - D'}(2y - x^2) = -\frac{1}{D'}\left(1 - \frac{D^2}{D'}\right)^{-1}(2y - x^2)$$

$$= -\frac{1}{D'}\left(1 + \frac{D^2}{D'} + \ldots.\right)(2y - x^2) = -\frac{1}{D'}(2y - x^2 - 2y)$$

$$= -\frac{1}{D'}(-x^2) = x^2 y$$

27. $$\frac{1}{D_1^3 + D_2^3 + D_3^3 - 3D_1D_2D_3}(-3xyz)$$

$$= -\frac{1}{3D_1D_2D_3}\left(1 - \frac{D_1^2}{3D_2D_3} - \frac{D_2^2}{3D_1D_3} - \frac{D_3^2}{3D_1D_2}\right)^{-1}(-3xyz)$$

$$= \frac{1}{D_1D_2D_3}\left(1 + \frac{D_1^2}{3D_2D_3} + \frac{D_2^2}{3D_1D_3} + \frac{D_3^2}{3D_1D_2} + \ldots.\right)(xyz)$$

$$= \frac{1}{D_1D_2D_3}(xyz) = \frac{x^2y^2z^2}{8}$$

29. $$\frac{1}{D^2 - 2DD' - 15D'^2}12xy$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D} - \frac{15D'^2}{D^2}\right)^{-1}12xy$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{15D'^2}{D^2} + \ldots\right)12xy$$

$$= \frac{1}{D^2}\left(12xy + 24\frac{1}{D}x\right) = \frac{12x^3y}{6} + \frac{24x^4}{4.6}$$

$$= 2x^3y + x^4$$

31. $$\frac{1}{D^2 - D'^2 - 3D + 3D'}e^{x+2y}$$

$$= \frac{1}{(D+D')\,(D-D') - 3\,(D-D')}e^{x+2y}$$

$$= \frac{1}{(D+D'-3)\,(D-D')}e^{x+2y}$$

$$= -\frac{1}{(D+D'-3)}e^{x+2y} = -e^{x+2y}\frac{1}{D+1+D'+2-3}.1$$

$$= -e^{x+2y}\frac{1}{D+D'}.1 = e^{x+2y}\frac{1}{D}\left(1 - \frac{D'}{D}\right)^{-1}.1$$

$$= -e^{x+2y}\frac{1}{D}.1 = -xe^{x+2y}$$

33. $$\left(x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2}\right) = xy$$

put $x = e^X$, $y = e^Y$ and $\frac{\partial}{\partial X} = D$, $\frac{\partial}{\partial Y} = D'$

we get $[D(D-1) - D'(D'-1)]\ z = e^{X+Y}$

$(D - D')\ (D + D' - 1)\ z = e^{X+Y}$

P.I. is $$\frac{1}{(D-D')\,(D+D'-1)}e^{X+Y} = \frac{1}{D-D'}e^{X+Y}$$

$$= e^{X+Y}\frac{1}{D+1+D'-1}.1 = e^{X+Y}\frac{1}{D+D'}.1$$

$$= e^{X+Y}\frac{1}{D}\left(1 + \frac{D'}{D}\right)^{-1}.1 = Xe^{X+Y} = xy\log x$$

34. Given $$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$$

put $x = e^X$, $y = e^Y$, $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$

we get $[D(D-1) - D'(D'-1) + D - D']z = X$

$(D^2 - D - D'^2 + D' + D - D')\ z = X$

or $(D - D')\ (D + D')\ z = X$

So P.I. is $$\frac{1}{(D^2 - D'^2)}.X = \frac{1}{D^2}\left(1 - \frac{D'^2}{D^2}\right)^{-1}.X$$

$$= \frac{1}{D^2}.X = \frac{X^3}{6} = \frac{(\log x)^3}{6}$$

36. If $z = \Sigma A e^{hx+ky}$ be the solution of $(D^2 - D')\ z = 0$

then it satisfies so

$$\Sigma A e^2 e^{hx+ky} - \Sigma kAe^{hx+ky} = 0$$

i.e. $h^2 = k$

38. Given $$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$$

i.e. $D^2 - a^2D'^2 = 0$

i.e. $(D + aD')\ (D - aD') = 0$

So. C.F. is $\phi_1(y + ax) + \phi_2(y - ax)$

39. Given $r + (a + b)\ s + abt = xy$

A.E. is $m^2 + (a + b)\ m + ab = 0$

i.e. $(m + a)\ (m + b) = 0$ or $m = -a, -b$

40. $$\frac{1}{D^2+D'^2}[-4\pi\,(x^2+y^2)]$$

$$= -\frac{4\pi}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{1}(x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left(1-\frac{D'^2}{D^2}+\ldots\right)(x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left(x^2+y^2-\frac{2}{D^2}.1\right) = -\frac{4\pi}{D^2}(x^2+y^2-x^2)$$

$$= -\frac{4\pi}{D^2}y^2 = -4\pi\frac{y^2x^2}{2} = -2\pi x^2y^2$$

43. $$\frac{1}{D^2-5DD'+4D'^2}\sin\,(4x+y)$$

$$\frac{1}{(D-4D')\,(D-D')}\sin\,(4x+y)$$

$$= \frac{-1}{3\,(D-4D')}\cos\,(4x+y)$$

$$= -\frac{x}{3}\cos\,(4x+y)$$

46. $$\frac{1}{D^2+3DD'+2D'^2}6\,(x+y) = \frac{6}{(1+3+2)}\,\frac{(x+y)^3}{3}$$

$$= \frac{(x+y)^3}{3}$$

49. $$\frac{1}{(D^2-2DD'+D'^2)}\sin\,(2x+3y)$$

$$= \frac{1}{(D-D')^2}\sin\,(2x+3y)$$

$$= \frac{1}{(2-3)^2}[-\sin\,(2x+3y)] = -\sin\,(2x+3y)$$

53. $$\frac{1}{(D^2-DD'+D'-1)}\cos\,(x+2y) = \frac{1}{D'}\cos\,(x+2y)$$

$$= \frac{D'}{D'^2}\cos\,(x+2y) = -\frac{1}{4}D'\cos\,(x+2y)$$

$$= +\frac{1}{2}\sin\,(x+2y)$$

55. $$\frac{1}{D^2-D'^2+D+3D'-2}x-y = -\frac{1}{4}e^{x-y}$$

58. Given $(D^2-DD'+D'-1)\ z=0$

or $(D-1)\,(D-D'+1)\ z=0$

So its solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x)$

61. $$\frac{1}{D^2-DD'+D'-1}e^y = \frac{1}{(D+1)\,(D-1)-D'(D-1)}e^y$$

$$= \frac{1}{(D-D'+1)\,(D-1)}e^y = \frac{-1}{D-D'+1}e^y$$

$$= -e^y\frac{1}{D-(D'+1)+1}.1 = -e^y\frac{1}{D-D'}.1$$

$$= -e^y\frac{1}{D}\left(1-\frac{D'}{D}\right)^{-1}.1 = -e^y.\frac{1}{D}.1 = -xe^y$$

64. $$\frac{1}{D^2-DD'+D'-1}\sin\,(x+2y) = \frac{1}{D'}\sin\,(x+2y)$$

$$= -\frac{D'}{D'^2}\sin\,(x+2y)$$

$$= -\frac{D'}{4}\sin\,(x+2y) = \frac{1}{2}\cos\,(x+2y)$$

66. If ΣAe^{hx+ky} be the solution of $(D^2-DD'+2D'-1)$ $z=0$ then it satisfied *i.e.*

$$(h^2-hx+2k-1)\ \Sigma Ae^{hx+ky} = 0$$

so $$h^2-hx+2k-1=0$$

$$k\,(2-h) = 1-h^2$$

or $$k = \frac{1-h^2}{2-h}$$

68. $$\frac{1}{D^2-D'}(2y-x) = -\frac{1}{D'}\left(1-\frac{D^2}{D'}\right)^{-1}(2y-x)$$

$$= -\frac{1}{D'}\left(1+\frac{D^2}{D'}+\ldots\right)(2y-x) = -\frac{1}{D'}(2y-x)$$

$$= -\,(y^2-xy) = xy-y^2$$

70. Given $x^2r^2-y^2+px-qy=(\log x)^2$

put $x=e^X$, $y=e^Y$, $D=\frac{\partial}{\partial X}$, $D'=\frac{\partial}{\partial Y}$ we get

$[D(D-1)-D'(D'-1)+D-D']\ z = X^2$

or $(D-D')\,(D+D')\ z = X^2$

P.I. is $\frac{1}{(D^2-D'^2)}-X^2$

$$= \frac{1}{D^2}\left(1-\frac{D'^2}{D^2}\right)^{-1}.X^2 = \frac{1}{D^2}X^2$$

$$= \frac{X^4}{12} = \frac{(\log x)^4}{12}$$

72. Given $s = e^{x+y}$

$$\Rightarrow \qquad \frac{\partial^2 z}{\partial x\partial y} = e^{x+y}$$

i.e. $$DD' = e^{x+y}$$

So, P.I. is $\frac{1}{DD'}e^{x+y} = e^{x+y}$

73. $$\frac{1}{D^3-D'^3}x^3y^3 = \frac{1}{D^3}\left(1-\frac{D'^3}{D^3}\right)^{-1}x^3y^3$$

$$= \frac{1}{D^3}\left(1+\frac{D'^3}{D^3}+\ldots.\right)x^3y^3$$

$$= \frac{1}{D^3}x^3y^3 + \frac{D'^3}{D^6}x^3y^3$$

$$= \frac{x^6y^3}{120} + \frac{1}{D^6}6x^3 = \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

75. $$\frac{1}{(D^3-4D^2D'+5DD'^2-2D'^3)}(y+x)^{1/2}$$
$$=\frac{1}{(D-D')^2(D-2D')}(y+x)^{1/2}$$
$$=-\frac{2}{3}\frac{1}{(D-D')^2}(y+x)^{3/2}$$
$$=-\frac{2}{3}\frac{x^2}{2!}(y+x)^{3/2}=-\frac{x^2}{3}(y+x)^{3/2}$$

76. P.I. is $\frac{1}{(D^2+D'^2)}.30\,(2x+y)$
$$=30\frac{1}{(4+1)}\frac{(2x+y)^3}{6}=(2x+y)^3$$

77. $$\frac{1}{D^2+D'^2}\cos mx\sin ny$$
$$=\frac{1}{2(D^2+D'^2)}\cos(mx+ny)+\frac{1}{2(D^2+D'^2)}\cos(mx-ny)$$
$$=\frac{-1}{2(m^2+n^2)}\cos(mx+ny)-\frac{1}{2(m^2+n^2)}\cos(mx-ny)$$
$$=-\frac{1}{2(m^2+n^2)}[\cos(mx+ny)+\cos(mx-ny)]$$

78. $$\frac{1}{(D^2-2DD'+D'^2)}x^3=\frac{1}{(D-D')^2}x^3$$
$$=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3=\frac{1}{D^2}\left(1+\frac{2D'}{D}+\ldots\right)x^3$$
$$=\frac{1}{D^2}x^3=\frac{x^5}{20}$$

79. $$\frac{1}{(D^2-4DD'+3D'^2)}(x+3y)^{1/2}$$
$$\frac{1}{1-4.1.3+3.9}(x+3y)^{5/2}\frac{2.2}{3.5}$$
$$=\frac{4}{16\times15}(x+3y)^{5/2}=\frac{(x+3y)^{5/2}}{60}$$

80. $$\frac{1}{D^2+6DD'+D'^2}e^x=\frac{1}{D^2+6DD'+D'^2}.e^{x+0y}$$
$$=\frac{1}{1+0+0}e^x=e^x$$

* * *

Unit-III

CHAPTER

10

Partial Differential Equations of Second Order with Variable Coefficients : Monge's Method

PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

1. If a partial differential equation contains at least one of the second order partial differential coefficients r, s, t but not of higher order is called second order PDE. Its general form is

$$f(x, y, z, p, q, r, s, t) = 0$$

After finding the general solution by usual method, the given geometrical conditions may be used to find arbitrary functions.

CLASSIFICATIONS OF SECOND ORDER PDE

1. Consider a linear PDE of second order in n-independent variables

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 \quad \text{...(1)}$$

Where a_{ij}, b_i, c are constants or function of $x_1, x_2, \ldots, x_n$

Let $\delta_i = \frac{\partial}{\partial x_i}$ and $\delta_i \delta_j = \frac{\partial^2}{\partial x_i \partial x_j}$

Consider, $\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \delta_i\, \delta_j$ ...(2)

Then partial differential equation (1) is

(i) Elliptic if ϕ is positive for all real values of δ_i and it reduced to zero only when all δ_i are zero.

(ii) Hyperbolic if ϕ can be both positive or negative.

(iii) Parabolic if the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

2. Consider a PDE of second order in two independent variables

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{...(1)}$$

Where A is positive.

Here $\phi = AS_1^2 + BS_1S_2 + CS_2^2$

So the equation (1) is

(i) Elliptic if $B^2 - 4AC < 0$

(ii) Hyperbolic if $B^2 - 4AC > 0$

(iii) Parabolic if $B^2 - 4AC = 0$

MONGE'S METHOD

1. Consider $Rr + Ss + Tt = V$ ...(1)

Where r, s, t have usual meaning and R, S, T, V are functions of x, y, z, p and q

Then we have $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = rdx + sdy$

and $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = sdx + tdy$

$\therefore\ r = \frac{dp - sdy}{ds}$ and $t = \frac{dq - sdx}{dy}$ put these values in equation (1) we get

$$(Rdpdy + Tdqdx - Vdxdy) - S(Rdy^2 - Sdxdy + Tdx^2) = 0$$

So the Monge's subsidiary equations are

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{...(2)}$$

and $dz = pdx + qdy$

It may be possible to obtain either one or two relations between x, y, z, p, q called intermediate integrals.

So we get general solution of (1).

2. Consider $Rr + Ss + Tt + U(rt - s^2) = V$

where, R, S, T, U, V are functions of x, y, z p and q

Then λ-quadratic equation is

$$\lambda^2(UV + RT) + \lambda SU + U^2 = 0$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The solution of $S = 0$ is :
   (a) $xf(x) + \phi(y)$ (b) $f(x) + y\phi(y)$
   (c) $xf(x) + y\phi(y)$ (d) $f(x) + \phi(y)$

2. $z = \log x \log y + f(x) + \phi(y)$ be the solution of :
   (a) $xyt = 1$ (b) $xyr = 1$
   (c) $xys = 1$ (d) $xyps = 1$

3. The solution of $t = \sin xy$ is :
   (a) $z = \frac{\sin xy}{x^2} + f(x) + \phi(y)$
   (b) $z = \frac{-\sin xy}{x^2} + yf(x) + \phi(x)$
   (c) $z = -\frac{\sin xy}{x} + yf(x) + \phi(x)$
   (d) $z = -\frac{\sin xy}{y} + yf(x) + \phi(x)$

4. The equation of surface satisfying $t = 6x^3y$ containing the two lines $y = z = 0$, $y = z = 1$ is :
   (a) x^3y^3 (b) $xy + y(1 - x^3)$
   (c) $xy + 1 - x^3$ (d) $x^3y^3 + y(1 - x^3)$

5. The solution of $xr = p$ is :
   (a) $f(y) + \phi(y)$ (b) $xf(y) + \phi(y)$
   (c) $x^2 f(y) + \phi(y)$ (d) $\frac{x^2}{2} f(y) + \phi(y)$

6. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is :
   (a) Parabolic (b) Hyperobolic
   (c) Elliptic (d) Circular

7. The equation $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ is :
   (a) Elliptic (b) Hyperbolic
   (c) Parabolic (d) None of these

8. If $z = \phi_1(y + x) + x\phi_2(y + x)$ be the general solution of $r + t - 2s = 0$ with $bz = y^2$ when $x = 0$, then $\phi_1(x + y)$ is :
   (a) $(x + y)$ (b) $b(x + y)^2$
   (c) $\frac{1}{2}(x + y)$ (d) $\frac{(x + y)^2}{b}$

9. The solution of $xr + Zp = 0$ is :
   (a) $-xf(y) + \phi(y)$ (b) $-\frac{1}{x} f(y) + \phi(y)$
   (c) $z = -f(y) + \phi(y)$ (d) $f(y) - y\phi(y)$

10. The equation $t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x\,\partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$ is hyperbolic if :
    (a) $tx = 1$ (b) $tx = 0$ (c) $tx > 1$ (d) $tx < 1$

11. The equation $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$ is parabolic if :
    (a) $x > 0$ (b) $x < 0$
    (c) $x = 0$ (d) For any value of x

12. The solution of $xr = (n - 1)\,p$ is :
    (a) $f(y) + x^n \phi(y)$ (b) $x^n + f(y) + \phi(y)$
    (c) $x^n f(y) + \phi(y)$ (d) $x^n f(x) + \phi(y)$

13. The Lagrange's equation for $t + s + q = 0$ is :
    (a) $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{f(x) + z}$ (b) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}$
    (c) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) + z}$ (d) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z - f(x)}$

14. The solution of $r = 2y^2$ is :
    (a) $z = x^2y^2 + xf(y) + \phi(y)$
    (b) $z = x^2y^2 + f(y) + y\phi(y)$
    (c) $z = x^2y^2 + f(y) + \phi(y)$
    (d) $x^2y^2 + xf(y) + y\phi(y)$

15. The solution of $t + s + q = 0$ is :
    (a) $e^x = f(x) + y\phi(y)$ (b) $ze^x = f(x) + y\phi(y)$
    (c) $ze^x = xf(x) + y\phi(y)$ (d) $ze^x = f(x) + \phi(x - y)$

16. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ is :
    (a) Circular (b) Hyperbolic
    (c) Parabolic (d) Elliptic

17. The solution of $r = 6x$ is :
    (a) $x^3 + xf(y) + \phi(y)$ (b) $x^3 + f(y) + y\phi(y)$
    (c) $x^2 + xf(y) + \phi(y)$ (d) $z = x^2 + xf(y) + y\phi(y)$

18. The equation $r = q$ is :
    (a) Circular (b) Elliptic
    (c) Parabolic (d) Hyperbolic

19. The equation $t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x\partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$ is parabolic if :
    (a) $tx > 1$ (b) $tx < 1$ (c) $tx = 0$ (d) $tx = 1$

20. If $z = \phi_1(2x + y) + x\phi_2(2x + y)$ be the general solution of $r - 4s + 4t = 0$ with $z = x = 0$ and $z - 1 = x - y = 0$ then $\phi_2(y + 2x)$ is equal to :

(a) $2x + y$
(b) $3(2x + y)$
(c) $-3(2x + y)$
(d) $\frac{3}{2x + y}$

21. The Lagrange's auxiliary equation for $p + r + s = 1$ is :

(a) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x - z + f(y)}$
(b) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + y + f(z)}$
(c) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x + y + f(z)}$
(d) $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x - z + f(y)}$

22. The Lagrange's auxiliary equation for $s - t = \frac{x}{y^2}$ is :

(a) $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{f(x) - \frac{x}{y}}$
(b) $\frac{dx}{-1} = \frac{dy}{1} = \frac{dz}{\frac{x}{y}}$
(c) $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{z + f(x)}$
(d) $\frac{dx}{-1} = \frac{dy}{1} = \frac{dz}{\frac{x}{y} - f(z)}$

23. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial u}{\partial t} = 0$ is :

(a) Elliptic
(b) Parabolic
(c) Hyperbolic
(d) None of these

24. The equation $x^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$ is hyperbolic if :

(a) $x > 0$
(b) $x < 0$
(c) $x = 0$
(d) For all value of x

25. The solution of $ar - xy = 0$ is :

(a) $az = \frac{x^2 y}{2} + f(y) + y\phi(y)$
(b) $az = \frac{x^3 y}{6} + xf(y) + \phi(y)$
(c) $az = xy + xf(y) + \phi(y)$
(d) $az = \frac{x^3 y}{6} + f(y) + \phi(y)$

26. The differential equation $\sqrt{x^2 + y^2}\frac{\partial^2 u}{\partial x^2} + 2(x - y)\frac{\partial^2 u}{\partial x \partial y} + \sqrt{x^2 + y^2}\frac{\partial^2 u}{\partial y^2} = 0$ in the second quadrant is :

(a) Circular
(b) Parabolic
(c) Hyperbolic
(d) Elliptic

27. The differential equation $xu_{tt} + tu_{xt} + u_{tt} = 0$ is hyperbolic if :

(a) $t^2 > 4x$
(b) $t^2 < 4x$
(c) $t^2 = 4x$
(d) None of thse

28. The solution of $2qy + y^2 t - 1 = 0$ is :

(a) $yz = xe^x - f(x) + y\phi(x)$
(b) $yz = e^x - f(x) + \phi(x)$
(c) $z = y\log y - f(x) + \phi(y)$
(d) $yz = y\log y - f(x) + y\phi(x)$

29. The solution of $q + xs - 4x - 2y = 2$ is :

(a) $2x^2 y + xy^2 + 2xy$
(b) $2x^2 y + xy^2 + 2xy + f(x) + \phi(y)$
(c) $2x^2 y + 2xy + f(x) + y\phi(y)$
(d) $2x^2 y + xy^2 + f(x) + \phi(y)$

30. In reducing the equation $r = x^2 t$ to canonical form choose u and v such that :

(a) $u = y + x, v = y - x$
(b) $u = y + \frac{x}{2}, v = y - \frac{x}{2}$
(c) $u = \frac{x^2}{2} + y, v = y - \frac{x^2}{2}$
(d) $u = x + \frac{y^2}{2}, v = x - \frac{y^2}{2}$

31. In reducing the equation $r + 2s + t = 0$ to canonical form choose u and v such that :

(a) $u = x - y, v = x + y$
(b) $u = x - \frac{y}{2}, v = x + \frac{y}{2}$
(c) $u = \frac{x}{2} - y, v = \frac{x}{2} + y$
(d) $u = \frac{x - y}{2}, v = \frac{x + y}{2}$

32. The equation $x^2 u_{tt} + 3u_{xt} + nu_{xx} + 17u_t - 100u = 0$ in third quadrant is :

(a) Parabolic
(b) Elliptic
(c) Circular
(d) Hyperbolic

33. The equation $x\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$ is parabolic if :

(a) $t^2 > 4x$
(b) $t^2 < 4x$
(c) $t^2 > 2x$
(d) $t^2 = 4x$

34. The solution of $S = e^{x+y}$ is :

(a) $e^{x+y} + f(y) + \phi(x)$
(b) $e^{x+y} + xf(y) + \phi(x)$
(c) $e^{x+y} + f(y) + x\phi(x)$
(d) $e^{x+y} + xf(y) + y\phi(x)$

35. In reducing the equation $r + x^2 t = 0$ to canonical form choose :

(a) $u = y + \frac{x^2}{2}, v = y - \frac{x^2}{2}$

(b) $u = y + ix\ v = y - ix$

(c) $u = y + ix^2, v = y - ix^2$

(d) $u = y + \frac{ix^2}{2}, v = y - \frac{ix^2}{2}$

36. In reducing the equation $x^2 r - 2xys + y^2 t - xp - 3yq - \frac{8y}{x} = 0$ to canonical form choose u and v such that :

(a) $u = xy, v = x - y$ (b) $u = xy, v = \frac{y}{x}$

(c) $u = x + y, v = x - y$ (d) $u = x + iy, v = x - iy$

37. To find Monge's subsidiary equation, r is equal to :

(a) $\frac{dq - sdx}{dy}$ (b) $\frac{dp - sdy}{dx}$

(c) $\frac{dp + sdy}{dx}$ (d) $\frac{dq + sdx}{dy}$

38. One of the Mong's subsidiary equation of $r = a^2 t$ is :

(a) $dpdy + a^2 dqdx = 0$ (b) $dy^2 + a^2 dx = 0$

(c) $dy^2 - a^2 dx = 0$ (d) $dp\,dy + a^2 dq = 0$

39. In reducing the equation $(y - 1)\, r - (y^2 - 1)\, s + y\,(y - 1)\, t + p - q = 2ye^{2x}(1 - y)^3$ to canonical form choose u and v such that :

(a) $u = x + y, v = ye^x$ (b) $u = x + y, v = x - y$

(c) $u = xe^y, v = ye^x$ (d) $u = xe^x, v = ye^y$

40. One of the Monge's subsidiary equation of $t - r\sec^4 y = 2q\tan y$ is :

(a) $dx - dy\tan y = 0$ (b) $dx - dy\sec^2 y = 0$

(c) $dx + dy\tan y = 0$ (d) $dx + dy\cos y = 0$

41. The Monge's subsidiary equation of $pt - qs = q^3$ is :

(a) $dx + pqdy = 0$ (b) $dy + pdx = 0$

(c) $pdx + qdy = 0$ (d) $pdx - qdy = 0$

42. Monge's subsidiary equation of $(r - s)\, y + (s - t)\, x + q - p = 0$ is :

(a) $xdy - ydx = 0$ (b) $xdx + ydy = 0$

(c) $dx + dy = 0$ (d) None of these

43. The Monge's subsidiary equation of $a^2 t(rt - s^2) = 0$ is :

(a) $dq + adx = 0$ (b) $dp + ady = 0$

(c) $dp - ady = 0$ (d) None of these

44. The equation $r + t = 0$ is :

(a) Parabolic (b) Hyperbolic

(c) Circular (d) Elliptic

45. $az = \frac{x^3 y}{6} + \frac{x^3}{3} + xf(y) + \phi(y)$ be the solution of :

(a) $r = 2axy$ (b) $ar = xy + 2x$

(c) $ar = xy + y$ (d) $ar = 2x + y$

46. The solution of $sy + p = \cos(x + y) - y\sin(x + y)$ is :

(a) $yz = y\sin(x + y) + f(x) + g(y)$

(b) $yz = \sin(x + y) + f(x) + g(y)$

(c) $yz = \cos(x + y) + f(x) + g(y)$

(d) None of these

47. To find Monge's subsidiary equation is equal to :

(a) $\frac{dq - sdx}{dy}$ (b) $\frac{dq + sdy}{dx}$

(c) $\frac{dq - sdy}{dx}$ (d) $\frac{dq + sdx}{dy}$

48. One of the Monge's subsidiary equation of $r - t\cos^2 x + p\tan x = 0$ is :

(a) $dx - \cos x dy = 0$ (b) $dy - \cos x dx = 0$

(c) $dy + \cos x dx = 0$ (d) $dx + \cos x dy = 0$

49. Monge's subsidiary equation of $pt - qs - q^3 = 0$ is :

(a) $pdq - q^3 dy = 0$ (b) $pdq - q^2 dy = 0$

(c) $pdq - qdy = 0$ (d) $pdx - qdy = 0$

50. Monge's subsidiary equation of $(r - s)\, y + (s - t)\, x + q - p = 0$ is :

(a) $dy - dx = 0$ (b) $xdy + ydx = 0$

(c) $ydy - xdx = 0$ (d) $xdx + ydy = 0$

51. Monge's subsidiary equation of $(q + 1)s - (p + 1)\, t = 0$ is :

(a) $pdq = 0$ (b) $(p - 1)\, dq = 0$

(c) $(q - 1)\, dy + (p - 1)\, dx = 0$

(d) $(p + 1)\, dq = 0$

52. The λ^- quadratic equaton of $rt - S^2 + 1 = 0$ is :

(a) $\lambda^2 + 1 = 0$ (b) $\lambda^2 - 1 = 0$

(c) $\lambda^2 + 2 = 0$ (d) $\lambda^2 - 2 = 0$

53. The λ^- quadratic equation for $2st\,(rt - s^2) = 1$ is :

(a) $\lambda^2 - 2\lambda + 1 = 0$ (b) $\lambda^2 - 1 = 0$

(c) $\lambda^2 + 2\lambda + 1 = 0$ (d) $\lambda^2 - 2\lambda - 1 = 0$

54. The λ^- quadratic equation for $5r + 6s + 3t + 2\,(rt - s^2) + 3 = 0$ is :

(a) $4\lambda^2 + 4\lambda + 1 = 0$ (b) $4\lambda^2 - 4\lambda + 1 = 0$

(c) $9\lambda^2 - 12\lambda + 4 = 0$ (d) $9\lambda^2 + 12\lambda + 4 = 0$

55. The Monge's subsidiary equation of $(x + y)(r - t) + 4p = 0$ is :

(a) $dy - dx = 0$ (b) $dy - xdy = 0$

(c) $ydy - dx = 0$ (d) $ydy - xdx = 0$

56. The Monge's subsidiary equation of $q^2r - 2pqs + p^2t = 0$ is :

(a) $ydx - xdy = 0$ (b) $qdpdx + pdqdy = 0$

(c) $xqdx - pydy = 0$ (d) $q^2dpdy + p^2dqdx = 0$

57. The Monge's subsidiary equation of $2s + (rt - s^2) = 1$ is :

(a) $dy + dp = 0$ (b) $dx - dq = 0$

(c) $xdx - dq = 0$ (d) $dx - qdq = 0$

58. The Monge's subsidiary equation of $r - 2s + t = \sin(2x + 3y)_{ss}$ is :

(a) $dy + dx = 0$ (b) $dy + xdx = 0$

(c) $dx + ydy = 0$ (d) $xdx + ydy = 0$

59. The λ quadratic equation of $r - x^2t = 0$ is :

(a) $\lambda^2 + x^2 = 0$ (b) $\lambda^2 - x^2 = 0$

(c) $\lambda^2 + x^2 + 2 = 0$ (d) None of these

60. The Lagrange's auxiliary equations for $r + s = 1$ is :

(a) $dx = dy = \dfrac{dz}{x + f(y)}$ (b) $dx = -ydy = \dfrac{dz}{f(y)}$

(c) $-dx = dy = \dfrac{dz}{x - f(y)}$ (d) $dx = dy = \dfrac{z}{x - (y)}$

61. The solution of $r = \sin xy$ is :

(a) $-\cos\dfrac{xy}{y^2} + yf(y) + \phi(x)$

(b) $-\dfrac{\sin xy}{y^2} + xf(y) + \phi(y)$

(c) $-\dfrac{\sin xy}{y^2} + yf(y) + \phi(x)$

(d) $-\cos\dfrac{xy}{y^2} + xf(y) + \phi(y)$

62. The solution of $t = xy$ is :

(a) $\dfrac{xy^3}{6} + f(x) + \phi(y)$ (b) $\dfrac{xy^3}{6} + xf(x) + \phi(y)$

(c) $\dfrac{xy^3}{6} + xf(x) + \phi(x)$ (d) $\dfrac{xy^3}{6} + yf(x) + \phi(x)$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(c)	**3.**	(b)	**4.**	(d)	**5.**	(d)	**6.**	(c)	**7.**	(b)	**8.**	(d)	**9.**	(b)	**10.**	(d)
11.	(c)	**12.**	(c)	**13.**	(b)	**14.**	(a)	**15.**	(d)	**16.**	(b)	**17.**	(a)	**18.**	(c)	**19.**	(d)	**20.**	(d)
21.	(d)	**22.**	(a)	**23.**	(b)	**24.**	(a)	**25.**	(b)	**26.**	(c)	**27.**	(a)	**28.**	(d)	**29.**	(b)	**30.**	(c)
31.	(a)	**32.**	(d)	**33.**	(d)	**34.**	(a)	**35.**	(d)	**36.**	(b)	**37.**	(b)	**38.**	(c)	**39.**	(a)	**40.**	b)
41.	(c)	**42.**	(c)	**43.**	(b)	**44.**	(d)	**45.**	(b)	**46.**	(a)	**47.**	(a)	**48.**	(b)	**49.**	(a)	**50.**	(c)
51.	(d)	**52.**	(b)	**53.**	(c)	**54.**	(d)	**55.**	(a)	**56.**	(d)	**57.**	(b)	**58.**	(a)	**59.**	(b)	**60.**	(a)
61.	(b)	**62.**	(d)																

HINTS AND SOLUTIONS

1. $$S = \frac{\partial^2 z}{\partial x \partial y} = 0$$

Integrating w.r.t. y we get $\dfrac{\partial z}{\partial x} = f(x)$

Again integrating w.r.t. x we get

$$z = \int f(x)dx + \phi(y)$$

or $$z = f(x) + \phi(y)$$

3. $$t = \frac{\partial^2 Z}{\partial y^2} = \sin xy$$

Integrating w.r.t.y, taking x as constant

$$\frac{\partial z}{\partial y} = -\frac{\cos xy}{x} + f(x)$$

Again integrating w.r.t. y we get

$$z = -\frac{\sin xy}{x^2}, + yf(x) + \psi(x)$$

4. Given that $t = 6x^3y$ with $y = z = 0$, $y = z = 1$

$i.e. \dfrac{\partial q}{\partial y} = 6x^3y$, integrating w.r.t. y we get

$$q = \frac{\partial z}{\partial y} = 3x^3y^2 + f(x)$$

Again integrating w.r.t. y,

$$z = x^3y^3 + yf(x) + \phi(x) \qquad \ldots(1)$$

Putting $y = 0$, $z = 0$, in (1) we get $\phi(x) = 0$

and putting $y = z = 1$ in (1) we get

$$1 \neq x^3 + f(x) + \phi(x)$$

$$\Rightarrow \quad f(x) = 1 - x^3$$

so by (1) we get the required surface *i.e.*,

$$z = x^3y^3 + y(1 - x^3)$$

6. Given $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

So $\phi = S_1^2 + S_2^2 + S_3^2$. Here ϕ is +ve for all real values of S_1, S_2, S_3 and it reduces to zero only when $S_1 = S_2 = S_3 = 0$. Hence it is elliptic.

7. Given equation is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Here, $A = C, B = 0, C = -1$

So, $B^2 - 4AC = 0 + 4C^2 = 4C^2 > 0$ so it is hyperbolic.

8. Given $r - 2s + t = 0$

$$\Rightarrow \quad (D^2 - 2DD' + D'^2)\, z = 0$$

$$(D - D')^2 z = 0$$

So its solution is

$$z = \phi_1(y + x) + x\phi_2(y + x)$$

Putting $x = 0$, $z = \dfrac{y^2}{b}$ we get

$$\frac{y^2}{b} = \phi_1(y) \text{ so } \phi(y + x) = \frac{(y + x)^2}{b}$$

10. Given differential equation is

$$t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

Here, $A = t, B = 2, C = x$

So $B^2 - 4AC = 4 - 4tx$

It is hyperbolic if $B^2 - 4AC > 0$

$$\Rightarrow \quad 4 - 4tx > 1$$

$$i.e. \quad tx < 1$$

12. Given differential equation is $\dfrac{\dfrac{\partial^2 z}{\partial x^2}}{\dfrac{\partial z}{\partial x}} = \dfrac{n-1}{x}$

Integrating we get $\log\dfrac{\partial z}{\partial x} = (n - 1)\log x + \log f(y)$

or $$\frac{\partial z}{\partial x} = x^{n-1}f(y)$$

Again integrating w.r.t. x we get

$$z = \frac{x^n}{n}f(y) + \psi(y) = x^n\phi(y) + \psi(y)$$

13. $t + s + q = 0$

$$i.e. \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = 0$$

$$\text{or} \quad \frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$$

Integrating w.r.t. y we get

$$q + p + z = f(x)$$

$$i.e., \quad p + q = f(x) - z$$

so Lagrange's equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f_{(x)} - z}$$

15. Given that $t + s + q = 0$

$$\text{or} \quad \frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$$

Integrating w.r.t. y $\;q + p + z = f(x)$

or $\quad p + q = f(x) - z$

So Lagrange's equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f_{(x)} - z}$$

By first and second member we get $x - y = 9$

By first and last members we get

$$\frac{dz}{dx} + z = f_{(x)}$$

So its I.F. is $e^{\int dx} = e^x$

Solution is $ze^x = \int e^x f(x)\, dx + b = \phi(x) + b$

$$i.e., \quad ze^x = \phi(x) + \psi(x - y)$$

16. Given that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} - \dfrac{1}{C^2}\dfrac{\partial^2 u}{\partial t^2} = 0$

$$\text{So} \quad \phi = S_1^2 + S_2^2 + S_3^2 - \frac{1}{C^2}S_4^2$$

So ϕ may be both positive or negative. Hence the equation is hyperbolic.

18. Given $r = q$

i.e., $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$$

Here, $A = 1, B = 0, C = 0$

So $B^2 - 4AC = 0 - 0 = 0$

so equation is parabolic.

20. Since $z = \phi_1(y+2x) + x\,\phi_2(y+2x)$ be the solution of given differerntial equation

put $z = x = 0$ we get $\phi_1(y+2x) = 0$

and put $z - 1 = x - y = 0$

we get $\phi_1(y+2x) + x\phi_2(y+2x) = 1$

Solving these $\phi_2(y+2x) = \frac{1}{x} = \frac{3}{3x} = \frac{3}{2x+y}$

22. Given $s - t = \frac{x}{y^2} \Rightarrow \frac{\partial p}{\partial y} - \frac{\partial q}{\partial y} = \frac{x}{y^2}$

Integrating w.r.t. y we get $p - q = -\frac{x}{y} + f(x)$

So its Lagrange's equation is

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-\frac{x}{y} + f(x)}$$

23. Given that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial u}{\partial t} = 0$

Comparing $\Sigma\Sigma a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \Sigma b_i \frac{\partial u}{\partial x_i} + cu = 0$

we get $a_{11} = a_{22} = a_{33} = 1,\ a_{44} = 0$

and all $a_{45} = 0\,\forall i \neq j$

So $$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Hence equation is parabolic.

26. Given $\sqrt{x^2+y^2}\,\frac{\partial^2 u}{\partial x^2} + 2\,(x-y)\,\frac{\partial^2 u}{\partial x \partial y}$

$$+\sqrt{x^2+y^2}\,\frac{\partial^2 u}{\partial y^2} = 0$$

Here $A = \sqrt{x^2+y^2}, B = 2(x-y), C = \sqrt{x^2+y^2}$

So, $B^2 - 4AC = (4x^2 + 4y^2 - 8xy) - (4x^2 + 4y^2)$

$= -8xy$

In second quadrant x is –ve & y is +ve

So $B^2 - 4AC > 0$

i.e. hyperbolic.

28. Given differential equation is $2yq + y^2 t = 1$

i.e. $$2y\frac{\partial z}{\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 1$$

Integrating w.r.t. y we get $y^2\frac{\partial z}{\partial y} = y + f(x)$

or $$\frac{\partial z}{\partial y} = \frac{1}{y} + \frac{f(x)}{y^2}$$

Again integrating w.r.t. y, $z = \log y - \frac{1}{y}f(x) + \phi(x)$

or $$yz = y\log y - f(x) + y\phi(x)$$

30. The given equation is $r - x^2 t = 0$

Comparing it with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

We have $R = 1, S = 0, T = -x^2$

So quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$\lambda^2 - x^2 = 0$$

$$\Rightarrow \qquad \lambda = x, -x$$

So we get $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

i.e. $\frac{dy}{dx} + x = 0$ and $\frac{dy}{dx} - x = 0$

Integrating $y + \frac{x^2}{2}$ = constant and $y - \frac{x^2}{2}$ = constant

So, $x = y + \frac{x^2}{2}$ and $v = y - \frac{x^2}{2}$

32. Given that $x^2 u_{tt} + 3u_x t + x4_{xx} + 17ut - 100\,u = 0$

Here, $A = x^2, B = 3, C = x$

So, $B^2 - 4AC = 9 - 4x^3$

In the third quadrant x is –ve so $B^2 - 4AC > 0$

i.e. it is hyperbolic.

34. $$S = \frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$$

Integrating w.r.t. y we get $\frac{\partial z}{\partial x} = e^{x+y} + f(y)$

Again integrating w.r.t x we get

$$z = e^{x+y} + xf(x) + f(y)$$

or $$z = e^{x+y} + \phi\,(x) + f(y)$$

35. Given that $r + x^2 t = 0$

Comparing with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

and Letting $R\lambda^2 + S\lambda + T = 0$ we get

$$\lambda^2 + x^2 = 0$$

$$\Rightarrow \qquad \lambda = \pm\, ix$$

so $\frac{dy}{dx} + ix = 0$ and $\frac{dy}{dx} - ix = 0$

or $$y+\frac{ix^2}{2}=\text{constant}$$

and $$y-\frac{ix^2}{2}=\text{constant}$$

So, $$u=y+\frac{ix^2}{2},\ v=y-\frac{ix^2}{2}$$

38. Given that $r=a^2t$

We know $dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=rdx+sdy$

$$dq=\frac{\partial q}{\partial x}dx+\frac{\partial q}{\partial y}dy=sdx+tdy$$

So, $r=\frac{dp-sdy}{dx}$ and $t=\frac{dq-sdx}{dy}$

Put these values in given equation we get

$$\frac{dp-sdy}{dx}=\left(\frac{dq-sdx}{dy}\right)a^2$$

or $(dpdy-a^2dqdx)-s\,(dy^2-a^2dx^2)=0$

So Monge's subsidiary equation is

$$dy^2-a^2dx^2=0$$

and $$dpdy-a^2dqdx=0$$

40. Put $r=\frac{dp-sdy}{dx}$ and $t=\frac{dq-sdx}{dy}$ in given equation

$t-r\sec^4 y=2q\tan y$ we have

$[dqdx-dpdy\sec^4 y-2q\tan y dxdy]$

$$-s\,[dx^2-dy^2\sec^4 y]$$

So Monge's subsidiary equation is $dx-dy\sec^2 y=0$

42. Put $r=\frac{dp-sdy}{dx}$ and $t=\frac{dq-sdx}{dy}$ in given equation

$$(r-s)\ y+(s-t)x+q-p=0$$

$$\left(\frac{dp-sdy}{dx}-s\right)y+\left(s-\frac{dq-sdx}{dy}\right)x+q-p=0$$

or $\{ydpdy-xdxdq+(q-p)\,dxdy\}-s\ \{ydy^2$

$$-(x-y)\,dxdy-xdx^2\}=0$$

So Monge's subsidiary equation is $dx+dy=0$

45. Consider $ar=xy+x$

$$\Rightarrow \qquad r=a\frac{\partial^2 z}{\partial x^2}=xy+2x$$

Integrating w.r.t. x we get

$$a\frac{\partial z}{\partial x}=\frac{yx^2}{2}+\frac{2x^2}{2}+f(y)$$

Again integrating we get

$$az=\frac{x^3y}{6}+\frac{x^3}{3}+xf(y)+\phi(y)$$

49. Given equation is $pt-qs=q^3$

put $t=\frac{dq-sdx}{dy}$ we get

$(pdq-q^3dy)-s\,(pdx+qdy)=0$

So Monge's subsidiary equation are $pdx+qdy=0$ and $pdq-q^3dy=0$

52. Given equation is $rt-s^2+1=0$

Comparing $Rr+Ss+Tt+U\ (rt-s^2)=V$ we have

$$R=0=S=T,U=1,V=-1$$

So λ^- quadratic equation is

$\lambda^2(UV+RT)+\lambda SU+U^2=0$

or $\lambda^2(-1)+\lambda\,(0)+1=0$

or $\lambda^2=1 \Rightarrow \lambda=1,-1$

54. Given $5r+6s+3t+2\,(rt-s^2)+3=0$

Comparing $Rr+Ss+Tt+U\ (rt-s^2)=V$

$$R=5,S=6,T=3,U=2,.\ V=-3$$

So λ^- quadratic equation is

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

or $$a\lambda^2+12\lambda+u=0$$

57. Given equation is $2s+(rt-s^2)=1$

Comparing with $Rr+Ss+Tt+U\ (rt-s^2)=V$

$R=0,S=2,T=0,U=1,V=1$

The λ quadratic equation is

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

or $\lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1$

So intermediate integral is

$$\lambda_2Rdy+Udx+\lambda_2Vdq=0$$

i.e. $$dx-dq=0$$

59. Given $r-x^2t=0$

Comparing $Rr+Ss+Tt+f=0$,

$$R=1,S=0,T=-x^2$$

So, λ-quadratic equation is

$R\lambda^2+S\lambda+T=0 \Rightarrow \lambda^2-x^2=0$

62. Given $t=\frac{\partial^2 z}{\partial y^2}=xy$

Integrating w.r.t. y, $\frac{\partial z}{\partial y}=\frac{xy^2}{2}+f(x)$

Again integrating w.r.t. y we get

$$z=\frac{xy^3}{6}+yf(x)+\phi(x)$$

* * *

Unit-III

CHAPTER 11

Legendre's Functions

LEGENDRE'S EQUATION

1. The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \text{ is called}$$

Lagendre's differential equation.

It can be writted

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0$$

where n is constant.

2. For solving

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \quad ...(1)$$

We assume that $y=\sum\limits_{r=0}^{\infty} a_r x^{k-r},\ a_0 \neq 0$

Differentiate it $\frac{dy}{dx}=\sum\limits_{r=0}^{\infty} a_r(k-r)x^{k-r-1}$

and $\frac{d^2y}{dx^2}=\sum\limits_{r=0}^{\infty} a_r(k-r)(k-r-1)\,x^{k-r-2}$

Put these values in equation (1) we get following cases :

Case I : When $k = n$ we have

$$y=a_0\left[x^4-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}+\ldots\right] \quad ...(2)$$

which is the first solution of Legendre's equation (1).

Case II : When $k = -(n+1)$ we have

$$y=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)\,x^{n-5}}{2.4(2n+3)(2n+5)}\right] ...(3)$$

which is the second solution of Legendre's equation (1).

3. The Legendre's function of first kind is defined as

$$P_n(x)=\frac{1.3.5\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}+\ldots\right]$$

It is also called Legendre's polynomial and may be denoted by

$$P_n(x)=\sum_{r=0}^{n/2}(-1)^r\frac{2n-2r!}{2^n r!\,n-2r!\,n-r!}x^{n-2r}$$

$$\text{where } \left(\frac{n}{2}\right)=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even}\\ \frac{n-1}{2} & \text{if } n \text{ is odd}\end{cases}$$

4. The Legendre's function of second kind *i.e.* Legendre's polynomial is

$$Q_n(x)=\frac{n!}{1.3.5\ldots.(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}\right]$$

5. The most general solution of the Legendre's equation is $y = AP_n(x) + BQ_n(x)$.

6. It may be shown that $P_n(x)$ is the coefficient of h^n in the expansion of $(1-2xh+h^2)^{-1/2}$ in ascending powers of h *i.e.*

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

where $\qquad P_0(x)=1$

Here, $(1-2xh+h^2)^{-1/2}$ is called the generating function of the Legendre polynomials.

7. The Laplace first integral for $P_n(x)$ is

$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\{x\pm\sqrt{(x^2-1)}\cos\phi\}^n d\phi$$

where n is a positive integer.

8. The Laplace second integral for $P_n(x)$ is

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{[x \pm \sqrt{(x^2-1)}\cos\phi]^{n+1}}$$

where n is a positive integer.

9. The orthogonal properties of Legendre's polynomials are

(i) $\int_{-1}^{+1} P_n(x)P_n(x)\,dx = 0$ if $m \neq n$

(ii) $\int_{-1}^{+1} [P_n(x)]^2\,dx = \dfrac{2}{2n+1}$

RECURRENCE FORMULAE

1. The Recurrence formulae are

(i) $(2n+1)\,xP_n = (n+1)\,P_{n+1} + nP_{n-1}$

or $nP_n = (2n-1)\,xP_{n-1} - (n-1)\,P_{n-2}$

(ii) $nP_n = xP'_n - P'_{n-1}$ where dashes denote differentiation w.r.t.x.

(iii) $(2n+1)\,P_n = P'_{n+1} - P'_{n-1}$

(iv) $(n+1)\,P_n = P'_{n+1} - xP'_n$

(v) $(1-x^2)P'_n = n\,(P_{n-1} - xP_n)$

(vi) $(1-x^2)\,P'_n = (n+1)\,(xP_n - P_{n-1})$

2. Beltrami's result is

$$(2n+1)\,(x^2-1)P'_n = n\,(n+1)\,(P_{n+1} - P_{n-1})$$

3. The Christoffel's Expension is

$$P'_n = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots.$$

the last term being $3P_1$ or P_0 according as n is even or odd.

4. The Christoffel's symmation formula is

$$\sum_{r=0}^{n} (2r+1)\,P_r(x)P_r(y) = (n+1)\,\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)\,P_n(x)}{(x-y)}$$

5. The Rodrigue's formula is

$$P_n(x) = \frac{1}{n!\,2^n}\frac{dn}{dx^n}(x^2-1)^n$$

6. The Murphy's formula is

$$P_n(x) = 1 - \frac{n\,(n+1)}{1.1}\left(\frac{1-x}{2}\right) + \frac{n\,(n-1)\,(n+2)\,(n+1)}{1.2.1.2}\left(\frac{1-x}{2}\right)^2 \ldots.$$

LEGENDRE'S POLYNOMIALS

Put $n = 0, 1, 2, 3, 4, \ldots$ respectively Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ we get}$$

$$P_0(x) = 1,\ P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x,$$

$$P_2(x) = \frac{3x^2-1}{2}$$

$$P_3(x) = \frac{5x^3-3x}{2},\ P_y(x) = \frac{35x^4 - 30x^2 + 3}{8} + \ldots$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. In the expansion of $(1-2xh+h^2)^{-1/2}$ in ascending power of h, $P_n(x)$ is the coefficient of :

(a) x^n (b) x^{-n} (c) h^n (d) h^{-n}

2. The value of $P_n(1)$ is :

(a) -1 (b) 1 (c) $(-1)^n$ (d) $(-1)^{2n}$

3. If n is odd then $P_n(0)$ is equal to :

(a) 0 (b) $(-1)^n$

(c) $\dfrac{(-1)^{n/2}n!}{2^n}$ (d) None of these

4. $P_0(x)$ is equal to :

(a) 1 (b) -1 (c) x (d) $-x$

5. The Legendre's differential equation is :

(a) $[(1+x^2)\,D^2 + 2xD + n]\,y = 0$

(b) $[(1-x^2)\,D^2 - 2xD + n]\,y = 0$

(c) $[(1-x^2)D^2 + 2xD + n\,(n+1)]\,y = 0$

(d) $[(1-x^2)\,D^2 - 2xD + n\,(n+1)]\,y = 0$

6. If $m \neq n$ then $\int_{-1}^{+1} P_m(x)P_n(x)\,dx$ is equal to :

(a) 0 (b) 1 (c) -1 (d) $\dfrac{2}{2n+1}$

7. $\int_{-1}^{1} P_1(x)^2 dx$ is equal to :

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 2 (d) $\frac{2}{3}$

8. The generating function of the Legendre's polynomial is :

(a) $(1-2xh+h^2)^{1/2}$ (b) $(1+2xh+h^2)^{-1/2}$

(c) $(1+2xh+h^2)^{1/2}$ (d) $(1-2xh+h^2)^{-1/2}$

9. The value of $P_n(-1)$ is :

(a) 1 (b) –1 (c) $(-1)^n$ (d) $(-1)^{2n}$

10. The value os $P_2(x)$ is :

(a) 1 (b) –1 (c) $\frac{2x-1}{3}$ (d) $\frac{3x^2-1}{2}$

11. The polynomial $x+3$ in terms of Legendre's polynomial is :

(a) $P_2(x)+3P_1(x)$ (b) $3P_1(x)+2$

(c) $2P_2(x)-2P_1(x)$ (d) $P_1(x)+3P_0(x)$

12. $P_n(x)=\frac{1}{2^n n!}\frac{dx}{dx^n}(x^2-1)^n$ is called :

(a) Recursion formula (b) Christophell's formula

(c) Legendre's formula (d) Rodrigue's formula

13. The value of $(n+1)\,P_{n+1}+nP_{n-1}$ is :

(a) $(n-1)\,xP_n$ (b) $(2n-1)\,xP_n$

(c) $(2n+1)\,xP_n$ (d) None of these

14. The value of $xP_n'-P_{n-1}'$ is :

(a) xP_n (b) nP_n (c) xP_{n+1} (d) nP_{n+1}

15. $\int_{-1}^{1} P_0(x)\,dx$ is equal to :

(a) 1 (b) 0 (c) 2 (d) –2

16. If a be the root of Legendre's polynomial $P_n(x)$ of degree n then $P_{n+1}(a)$ and $P_{n-1}(a)$ are of :

(a) Same signs (b) Opposite signs

(c) Both zero (d) None of these

17. The value of $P_n'(1)$ is :

(a) n (b) $\frac{n(n+1)}{2}$

(c) $\frac{n\,(n-1)}{2}$ (d) n^2-1

18. If n is even then $P_n(0)$ is equal to :

(a) $(-1)^n$ (b) $(-1)^n\frac{n\,(n+1)}{2}$

(c) $\frac{(-1)^{n/2} n!}{2^n (n/2!)^2}$ (d) $\frac{(-1)^n n!}{\left(\frac{n}{2}\right)!^2}$

19. If $m=n$ then $\int_{-1}^{+1} P_n(x)^2 dx$ is equal to :

(a) $\frac{1}{2n+1}$ (b) $\frac{1}{2n-1}$

(c) $\frac{2}{2n+1}$ (d) $\frac{2}{2n-1}$

20. The value of $(n+1)\,(nxP_n-P_{n+1})$ is :

(a) $(1+x^2)P_n$ (b) $(1-x)\,P_n$

(c) $(1+x)\,P_n$ (d) $(1-x^2)\,P_n'$

21. $\int_{-1}^{1} P_n(x)\,dx$, where $n\neq 0$ is equal to :

(a) 0 (b) 1 (c) 2 (d) $\frac{3}{2}$

22. If $m<n$ then $\int_{-1}^{1} x^m P_n(x)\,dx$ is equal to :

(a) x (b) x^2-1 (c) $\frac{1}{m+n}$ (d) 0

23. All the roots of $P_n(x)=0$ are :

(a) Same (b) Distinct

(c) Two roots are same (d) Zero

24. $\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx$ is equal to :

(a) $\frac{2}{2n^2+1}$ (b) $\frac{2}{4n^2+1}$

(c) $\frac{2}{4n^2-1}$ (d) $\frac{2}{2n^2-1}$

25. The value of $\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$ is equal to :

(a) $P_n(x)$ (b) $P_n(-x)$ (c) $P_{-n}(x)$ (d) 0

26. The value of $P_n'(-1)$ is :

(a) $(-1)^n$ (b) $\frac{(-1)^n(n+1)}{2}$

(c) $\frac{(-1)^{n-1}n\,(n+1)}{2}$ (d) $\frac{(-1)^n n\,(n+1)}{2}$

27. The value of $P_{n+1}'-P_{n-1}'$ is :

(a) nP_n (b) $(2n-1)\,P_n$

(c) $(2n-1)\,P_{n-1}$ (d) $(2n+1)\,P_n$

28. All the roots of $P_n(x)=0$ are :

(a) Real only

(b) Complex only

(c) Real and complex both

(d) None of these

29. The polynomial x^2+x+1 in terms of Legendre's polynomial :

(a) $P_2(x)+\frac{2}{3}P_1(x)+\frac{1}{3}P_0(x)$

(b) $P_2(x)+\frac{1}{3}P_0(x)$

(c) $\frac{2}{3}P_2(x)+P_1(x)+\frac{4}{3}P_0(x)$

(d) $\frac{2}{3}P_2x+\frac{1}{3}P_0(x)$

30. The value of $\int_{-1}^{1} xP_1(x)P_0(x)\,dx$ is :
(a) 0 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 1

31. $\int_{-1}^{1}(1-x^2)\,P_m'P_n'dx$, where m, n are distinct the integers is equal to :
(a) 0 (b) $\frac{m+n}{mn}$ (c) m^2+n (d) $\frac{m-n}{m^2n^2}$

32. The value of $\int_{-1}^{1} xP_2(x)\,dx$ is equal to :
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 0

33. The value of $\int_{-1}^{1}[P_0(x)]^2 dx$ is equal to :
(a) 0 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{3}{2}$

34. The value of $\int_{-1}^{1} P_1(x)P_2(x)\,dx$ is equal to :
(a) $\frac{3}{2}$ (b) 0 (c) 2 (d) $\frac{2}{3}$

35. The value of $P_1(x)$ is :
(a) $\frac{3x^2-1}{2}$ (b) $2x-1$
(c) x (d) 1

36. The value of $P_{n+1}'-xP_n'$ is :
(a) nP_n (b) $(n+1)\,P_n$
(c) $(n-1)\,P_n$ (d) nP_{n-1}

37. The roots of $P_n(x)=0$ lies between :
(a) 0 to 1 (b) -1 to $+1$
(c) $-\infty$ to $+\infty$ (d) All zeros

38. The polynomial x^2+1 in terms of Legendre polynomial is :
(a) $\frac{3}{2}P_{2(x)}+P_0(x)$ (b) $\frac{2}{3}P_2(x)+3P(x)$
(c) $\frac{2}{3}P_2(x)+P_1(x)+\frac{4}{3}P_0(x)$
(d) $\frac{2}{3}P_2(x)+\frac{4}{3}P_0(x)$

39. The value of $\int_{-1}^{1} x\,P_1(x)P_2(x)$ is :
(a) 0 (b) $\frac{3}{10}$ (c) $\frac{4}{13}$ (d) $\frac{4}{15}$

40. The value of $\int_{-1}^{1}(1-x^2)P_1'P_2'\,dx$ is :
(a) 0 (b) 1 (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

41. The value of $P_3(1)$ is equal to :
(a) 1 (b) -1 (c) 5 (d) 3

42. The value of $n(P_{n-1}-xP_n)$ is :
(a) $(1-x^2)P_n'$ (b) $(1+x^2)P_n$
(c) $(1-x^2)\,P_n$ (d) None of these

43. The polynomial x^3 in terms of Legendre polynomial is :
(a) $\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$
(b) $\frac{1}{5}P_3(x)+\frac{2}{5}P_2(x)+P(x)$
(c) $\frac{1}{3}P_3(x)+\frac{3}{5}P_2(x)$ (d) $\frac{2}{5}P_3(x)+\frac{2}{5}P_2(x)$

44. The value of $\int_{-1}^{1} x^2P_3(x)\,dx$ is equal to :
(a) 1 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) 0

45. The value of $\int_{-1}^{1} P_2(x)P_3(x)\,dx$ is equal to :
(a) $\frac{5}{6}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) 0

46. $2P_2+P_0$ is equal to :
(a) xP_1 (b) $3xP_1$ (c) $3P_1$ (d) $3x$

47. The value of $xP_1'-P_0'$ is :
(a) P_0 (b) P_1 (c) P_2 (d) xP_1

48. The value of $P_2'-xP_1'$ is
(a) P_1 (b) P_2 (c) $2P_1$ (d) $2P_2$

49. The value of $\int_{-1}^{1}(P_1')^2\,dx$ is :
(a) 0 (b) 1 (c) 2 (d) 3

50. The expansion of $P_n(x)$ in powers of $\frac{1-x}{2}$ is called :
(a) Legendre formula
(b) Murphy's formula
(c) Recurrence formula
(d) Christoffel's summication formula

51. $P_8'+9P_9$ is equal to :
(a) xP_9' (b) $(x+1)\,P_9$
(c) $P_9'-x$ (d) $P_9'+x$

52. The value of $\int_{-1}^{1}(1-x^2)\,(P_0')^2\,dx$ is :
(a) 0 (b) 1 (c) 2 (d) 3

53. $\int_{-1}^{1} x^2P_2(x)dx$ is equal to :
(a) $\frac{4}{3}$ (b) $\frac{4}{7}$ (c) $\frac{4}{15}$ (d) $\frac{4}{13}$

54. $P_2'(1)$ is equal to :
(a) 0 (b) 1 (c) 2 (d) 3

55. If $-1 \le x \le 1$ then :

(a) $0 \le P_n(x) \le 1$ (b) $-\infty \le P_n(x) < \infty$

(c) $|P_n(x)| \le 1$ (d) $1 \le |P_n(x)|$

56. If $P_n(x)$ and $Q_n(x)$ are Legendre's function of first and second kind the general solution of Legendre's equation is :

(a) $AP_n(x)$ (b) $BQ_n(x)$

(c) $AP_n(x) + BQ_n(x)$ (d) $AP_n(x)Q_n(x)$

57. $P_n(x) = \frac{1}{\pi}\int_0^\pi [x \pm \sqrt{(x^2-1)}\cos\phi]^n \, d\phi$, where n is positive integer is called :

(a) Recurrence formula (b) Laplace first integral

(c) Beltrami's result

(d) Laplace second integral

58. $P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x \pm \sqrt{(x^2-1)}\cos\phi]^{n+1}}$, where n is positive integer is called :

(a) Laplace first integral (b) Legendre first integral

(c) Laplace second integral

(d) Christoffel's formula

59. $P_3' - xP_2'$ is equal to :

(a) $2P_3 + P_2$ (b) $3P_2$

(c) $3P_1$ (d) $3P_0$

60. $P_3' - P_1'$ is equal to :

(a) $5P_2$ (b) $3P_2$

(c) $5P_0$ (d) $3P_0$

61. The value of $P_n'(0)$ is :

(a) $nP_n(0)$ (b) $nP_{n-1}(0)$

(c) $(n+1)\,P_n(0)$ (d) None of these

62. The value of $P_n'(0)$ is :

(a) nP_{n+1} (b) $(n+1)\,P_{n+1}(0)$

(c) $-(n+1)\,P_{n+1}$ (d) $nP_n(0)$

63. $P_5(0)$ is equal to :

(a) 0 (b) $\frac{2}{3}$ (c) $\frac{5}{3}$ (d) $\frac{1}{2}$

64. $P_4(0)$ is equal to :

(a) 0 (b) $\frac{2}{3}$ (c) $\frac{5}{8}$ (d) $\frac{3}{8}$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(a)	**5.**	(d)	**6.**	(a)	**7.**	(d)	**8.**	(d)	**9.**	(c)	**10.**	(d)
11.	(d)	**12.**	(d)	**13.**	(c)	**14.**	(b)	**15.**	(c)	**16.**	(b)	**17.**	(b)	**18.**	(c)	**19.**	(c)	**20.**	(d)
21.	(a)	**22.**	(d)	**23.**	(b)	**24.**	(c)	**25.**	(a)	**26.**	(c)	**27.**	(d)	**28.**	(a)	**29.**	(c)	**30.**	(b)
31.	(a)	**32.**	(d)	**33.**	(c)	**34.**	(b)	**35.**	(c)	**36.**	(b)	**37.**	(b)	**38.**	(d)	**39.**	(d)	**40.**	(a)
41.	(a)	**42.**	(a)	**43.**	(a)	**44.**	(d)	**45.**	(d)	**46.**	(b)	**47.**	(b)	**48.**	(c)	**49.**	(c)	**50.**	(b)
51.	(a)	**52.**	(a)	**53.**	(c)	**54.**	(d)	**55.**	(c)	**56.**	(c)	**57.**	(b)	**58.**	(c)	**59.**	(b)	**60.**	(a)
61.	(b)	**62.**	(c)	**63.**	(a)	**64.**	(d)												

HINTS AND SOLUTIONS

2. Since $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$

Put $x = 1$, $\sum_{n=0}^{\infty} h^n P_n(1) = (1 - 2h + h^2)^{-1/2} = (1-h)^{-1}$

$$= 1 + h + h^2 + \ldots + = \sum_{n=0}^{\infty} h^n$$

Equating the coefficient of h^n on both sides we get

$$P_n(1) = 1$$

3. Since $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$

Put $x = 0$ $\Sigma h^n P_n(0) = (1 + h^2)^{-1/2}$

$$= [1 - (-h^2)]^{-1/2}$$

$$= 1 + \frac{1}{2}(-h^2) + \frac{1.3}{2.x}(-h^2)^2 + \ldots + \frac{1.3.5\ldots(2r-1)}{2.4\ldots 2r}(-h^2)^r + \ldots$$

Equating the coefficient of h^4 on both sides we have

$$P_n(0) = 0 \text{ if } n \text{ is odd.}$$

7. $\because \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2x+1}$

when $m = n$, put $m = n = 1$

So $\int_{-1}^{1} [P_1(x)]^2 dx = \frac{2}{2+1} = \frac{2}{3}$

9. $\because \Sigma h^n P_n(x) = (1-2xh+h^2)^{-1/2}$, put $h \to -h$

$\Sigma(-1)^n h^n P_n(x) = (1+2xh+h^2)^{-1/2}$ ...(1)

Again put $x \to -x$ in first equation

$\Sigma h^n P_n(-x) = (1+2xh+h^2)^{-\frac{1}{2}}$ ...(2)

But equation (1) and (2)

$\Sigma h^n P_n(-x) = \Sigma(-1)^n h^n P_n(x)$

Equating coefficient of h^n on both sides

$P_n(-x) = (-1)^n P_n(x)$

$\Rightarrow \quad P_n(-1) = (-1)^n$

10. Rodrigue's formula is

$$P_n(x) = \frac{1}{2^n n!}\frac{dx}{dx^n}(x^2-1)^n$$

So, $P_2(x) = \frac{1}{2^2\, 2!}\frac{d^2}{dx^2}(x^2-1)^2$

$$= \frac{1}{8}\frac{d}{dx}[4x(x^2-1)]$$

$$= \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1)$$

11. $\because P_0(x) = 1, P_1(x) = x$

So, $f(x) = x + 3 = P_1(x) + 3P_0(x)$

15. $\int_{-1}^{1} P_0(x)dx = \int_{-1}^{1} 1\,dx = 2$

16. By recurrance formula I

$(2x+1)\,xP_n(x) = (n+1)\,P_{n+1}(x) + nP_{n-1}(x)$

Given a is root of $P_n(x)$ so $P_n(a) = 0$.

Put in first equation $(x = a)$

$(2n+1)\,a.0 = (n+1)\,P_{n+1}(a) + nP_{n-1}(a)$

or $\frac{P_{n+1}(a)}{P_{n-1}(a)} = \frac{-n}{n+1}$

Since n is +ve integer so R.H.S. is negative *i.e.*, $P_{n+1}(a)$ and $P_{n-1}(a)$ must be of opposite signs.

18. Since $\sum_{n=0}^{\infty} h^n P_n(x) = \left(1-2xh+h^2\right)^{-\frac{1}{2}}$

Put $x = 0$, $\Sigma h^n P_n(0) = (1+h^2)^{-\frac{1}{2}} = [1-(-h^2)]^{-\frac{1}{2}}$

$= 1 + \frac{(-h)^2}{2} + \frac{1.3}{2.4}(-h^2)^2 + \ldots \frac{1.3.5\ldots(2r-1)\,(h^2)^r}{2.4..2r}$

Equating the coefficient of h^{2m} on both sides we get

$$P_{2m}(0) = \frac{1.3.5....(2m-1)\,(-1)^m}{2.4.6.....2m} = (-1)^m \frac{2m!}{2^{2m}(m!)^2}$$

Put $n = 2m$ we get

$$P_n(r) = \frac{(-1)^{n/2} n!}{2^n\left(\frac{n}{2}!\right)^2}.$$

20. This is a recurrence formula.

21. By Rodrigue's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Integrating from -1 to $+1$

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} = 0$$

24. From Recurrence formula I

$$xP_n(x) = \frac{(n+1)\,P_{n+1}(x) + nP_{n-1}(x)}{2n+1}$$

$\therefore \int_{1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{1}{2n+1}$

$$\int_{-1}^{1}[(n+1)\,P_{n+1}(x) + nP_{n-1}(x)]P_{n-1}(x)\,dx$$

$$= \frac{1}{2n+1}[(n+1)\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)\,dx$$

$$+ n\int_{-1}^{1}[P_{n-1}(x)^2\,dx]$$

$$= \frac{1}{2n+1}\left[0 + n\frac{2}{2\,(n-1)+1}\right] = \frac{2n}{4n^2-1}$$

26. Since $P_n(x)$ is the solution of Legendre's differential equation

so, $(1-x^2)P_n''(x) - 2xP_n'(x) + n\,(n+1)\,P_n(x) = 0$

Put $x = -1$ we get,

$2P_n'(-1) + n\,(n+1)\,P_n(-1) = 0$

$P_n'(-1) = -\frac{1}{2}n(n+1)\,P_n(-1) = -\frac{1}{2}n\,(n+1)\,(-1)^n$

$= (-1)^{n-1}\frac{n}{2}(n+1)$

29. Given polynomial $f(x) = x^2 + x + 1$

and we know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3x^2-1}{2}$

So, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

$\therefore f(x) = x^2 + x + 1 = \frac{2}{3}P_2(x) + \frac{1}{3} + P_1(x) + 1$

or $f(x) = \frac{2}{3}P_2(x) + P_1(x) + \frac{4}{3}P_0(x)$

31. $\int_{-1}^{1}(1-x^2)P'_mP'_n dx = [(1-x^2)P'_mP_n]_{-1}^{1}$

$$-\int_{-1}^{1}P_n\frac{d}{dx}[(1-x^2)P'_m]\,dx$$

$$= -\int_{-1}^{1}\left[P_n\frac{d}{dx}\{(1-x^2)P'_m\}\right] \qquad ...(1)$$

By Legendre's equation $\frac{d}{dx}\{(1-x^2)P'_m\} = -m\,(m+1)\,P_m$ so by equ. (1)

$$\int_{-1}^{1}(1-x^2)P'_mP'_n dx = -\int_{-1}^{1}[-P_n m(m+1)\,P_m]\,dx$$

$$= m\,(m+1)\int_{-1}^{1}P_nP_m dx = 0 \qquad (\because m \neq n)$$

33. Put $P_0(x) = 1$ we get required result.

36. By Recurrence formula $P'_{n+1} - xP'_n = (n+1)P_n$

38. Given polynomial is $f(x) = x^2 + 1$

$\because \quad P_0(x) = 1, P_2(x) = \frac{1}{2}(3x^2 - 1)$

So, $\quad 3x^2 - 1 = 2P_2(x)$

$\Rightarrow \quad x^2 = \frac{2P_2(x)+1}{3}$

So, $f(x) = \frac{2P_2(x)+1}{3} + 1 = \frac{2P_2(x)}{3} + \frac{4}{3}$

$$= \frac{2}{3}P_2(x) + \frac{4}{3}P_0(x)$$

39. $\because \int_{-1}^{1}xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$

So put $n = 2$ we get

$$\int_{-1}^{1}xP_2P_n(x) = \frac{4}{16-1} = \frac{4}{15}$$

41. $\because P_3(x) = \frac{5x^3 - 3x}{2}$ by Rodrigue's formula

So, $\quad P_3(1) = \frac{5-3}{2} = 1$

43. $\because P_3(x) = \frac{5x^3 - 3x}{2}$,

$$P_2(x) = \frac{3x^2 - 1}{2}, P_1(x) = x, P_0(x) = 1$$

So, $\quad x^3 = \frac{2P_3 + 3x}{5} = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

44. Put $P_3(x) = \frac{5x^3 - 3x}{2}$ and then integrating we get the required result.

45. By orthogonal property

$\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$ if $m \neq n$

Put $m = 2, n = 3$ we get

$\int_{-1}^{1}P_2(x)P_3(x)dx = 0$

48. By Recurrence formula $P'_{n+1} - xP'_n = (n+1)\,P_n$

Put $n = 1$ we get $P'_2 - xP'_1 = 2P_1$

51. By recurrence formula $nP_n = xP'_n - P'_{n-1}$

Put $n = 9$, $9P_9 = xP'_9 - P'_8$

i.e. $\quad P'_8 + 9P_9 = xP'_9$

54. $\because P_2(x) = \frac{3x^2 - 1}{2}$

$\Rightarrow \quad P'_2(x) = 3x$

$\therefore \quad P'_2(1) = 3$

59. By Recurrence formula

$P'_{n+1} - xP'_n = (n+1)\,P_n$

Put $\quad n = 2$ we get

$P'_3 - xP'_2 = 3P_2$

60. By Recurrence formula $P'_{n+1} - P'_{n-1} = (2n+1)\,P_n$

Put $n = 2$ we get $P'_3 - P'_1 = 5P_2$

61. By Recurrence formula $(1-x^2)P'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$

Put $x = 0$ we get $P'_n(0) = nP_{n-1}(0)$

62. By Recurrence formula

$(1-x^2)P'_n = (n+1)\,(xP_n - P_{n+1})$

Put $x = 0$ we get $P'_n(0) = -(n+1)\,P_{n+1}(0)$

63. Since $P_n(0) = 0$ for all odd value of n so $P_5(0) = 0$

64. Since $P_n(0) = \frac{(-1)^{n/2}n!}{2^n\left(\frac{n}{2}!\right)^2}$ for n is even.

So $\quad P_4(0) = \frac{n!}{16(2)^2} = \frac{4.3.2}{16\times 4} = \frac{3}{8}$

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Unit-III

CHAPTER 12

Bessel's Functions

BESSEL'S EQUATION

1. The differential equation of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

is called Bessel's differential equation.

2. Consider the Bessel's differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \qquad \ldots(1)$$

Assume that its series solution is $y = \sum_{r=0}^{\infty} a_r x^{k+r}$

then $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)\,x^{k+r-1}$

and $\frac{d^2y}{dx^2} = \Sigma a_r(k+r)\,(k+r-1)x^{k+r-2}$

Putting these values in equation (1) we get

$$\Sigma a_r[\{(k+r)^2 - n^2\}\,x^{k+r-2} + x^{k+r}] = 0$$

Case I : If $k = n$ we get

$$y = \frac{x^n}{2n!(n+1)}\left[1 + (-1)\frac{x^2}{2^2 1!(n+1)} + (-1)^2\frac{x^4}{2^4 2!(n+1)(n+2)} + \ldots\right]$$

This is known as Bessel's function of the first kind of order n.

So, $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!+(n+r+1)}$

Case II : If $k = -n$ we get

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{r!+(-n+r+1)}$$

The solution of Bessel's equation are called Bessel's functions.

3. When n is not an integer the most general solution of Bessel's equation is

$$y = AJ_{-n}(x) + BJ_n(x)$$

where A and B are arbitrary constants.

RECURRENCE FORMULAE OF $J_n(x)$

1. The recurrence formulae of $J_n(x)$ are

(i) $xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$

(ii) $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)$

or $\frac{d}{dx}(x^nJ_n) = x^nJ_{n-1}$

(iii) $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

(v) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

GENERATING FUNCTION FOR $J_n(x)$

1. When n is a positive integer, $J_n(x)$ is the coefficient of Z^n in the expansion of $\dfrac{e^x\left[z - \frac{1}{2}\right]}{2}$ in ascending and descending powers of z.

2. $J_n(x)$ is the coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of $e^{x\left[z-\frac{1}{z}\right]/2}$.

Thus, $e^{\frac{x\left(z-\frac{1}{2}\right)}{2}} = \sum_{-\infty}^{\infty} z^nJ_n(x)$

ORTHOGONAL PROPERTIES

$$\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = \begin{cases} 0 & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2 & \alpha = \beta \end{cases}$$

where α, β are the roots of $J_n(x) = 0$.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The Bessel's equation is :

(a) $\frac{d^2y}{dx^2}+\frac{xdy}{dx}+x^2y=0$

(b) $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$

(c) $\frac{d^2y}{dx^2}+\frac{xdy}{dx}-\frac{n^2}{x^2}y=0$ (d) None of these

2. If n is a positive integer then $J_n(x)$ is equal to :

(a) $J_{-n}(x)$ (b) $J_n(-x)$

(c) $(-1)^nJ_n(x)$ (d) None of these

3. If n is even then $J_n(x)$ is :

(a) Odd (b) Even

(c) May be odd or even (d) None of these

4. $J_n(-x)$ is equal to :

(a) $J_{-n}(x)$ (b) $-J_n(x)$

(c) $(-1)^nJ_n(x)$ (d) $(-1)^{n+1}J_n(x)$

5. $nJ_n(x)-xJ_{n+1}(x)$ is equal to:

(a) $J_n'(x)$ (b) $-xJ_n'(x)$

(c) $-J_n'(x)$ (d) $xJ_n'(x)$

6. $J_{n-1}-J_{n+1}$ is equal to :

(a) $2J_n$ (b) $2J_n'$ (c) $2J_{n+1}$ (d) J_{n-1}'

7. $J_{\frac{1}{2}}(x)$ is equal to :

(a) $\sqrt{\left(\frac{\pi x}{2}\right)}\sin x$ (b) $\sqrt{\left(\frac{2}{\pi x}\right)}\cos x$

(c) $\sqrt{\frac{\pi x}{2}}\cos x$ (d) $\sqrt{\left(\frac{2}{\pi x}\right)}\sin x$

8. $-nJ_n+xJ_{n-1}$ is equal to :

(a) $xJ_n(x)$ (b) $-xJ_n(x)$

(c) $xJ_n'(x)$ (d) $-xJ_n'(x)$

9. The value of $[J_{1/2}(x)]^2+[J_{-1/2}(x)]^2$ is :

(a) 0 (b) 1

(c) $\frac{2}{\pi x}$ (d) $\frac{\pi x}{2}$

10. $J_{-n}(-x)$ is equal to :

(a) $J_n(-x)$ (b) $-J_n(x)$

(c) $(-1)^nJ_n(x)$ (d) $(-1)^{n-1}J_n(x)$

11. The coefficient of z^n in the expansion of $e^{\frac{x\left[z-\frac{1}{z}\right]}{2}}$ is :

(a) $P_n(x)$ (b) $P_n'(x)$ (c) $J_n(x)$ (d) $J_n'(x)$

12. If α,β are the roots of $J_n(x)=0$ and $\alpha\neq\beta$ then $\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx$ is :

(a) 0 (b) 1

(c) $[J_{n+1}(\alpha)]^2$ (d) $J_n(\alpha)+J_n(\beta)$

13. The value of $\int_0^a xJ_n^2(\alpha x)\,dx$ where α is roots of $J_n(x)=0$:

(a) 0 (b) $\frac{a^2}{2}J^2{}_{n+1}(a\alpha)$

(c) $aJ_{n+1}(a\alpha)$ (d) $\frac{J_{n+1}^2(a\alpha)}{2}$

14. The most general solution of Bessel's equation is :

(a) $AJ_n(x)$ (b) $AJ_{-n}(x)$

(c) $AJ_n(x)+BJ_{-n}(x)$ (d) $AJ_n(-x)+BJ_n(x)$

15. The solution of the differential equation $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$ is :

(a) $J_n(x)$ (b) $J_1(x)$ (c) $J_2(x)$ (d) $J_0(x)$

16. $\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r r!)^2}$ is equal to :

(a) $J_0(x)$ (b) $J_1(x)$ (c) $J_n(x)$ (d) $J_r(x)$

17. If n is odd then $J_n(x)$ is:

(a) Even (b) Odd

(c) May be even or odd (d) 0

18. $\frac{d}{dx}(x^{-n}J_n)$ is equal to :

(a) $-x^{-n}J_{n+1}$ (b) $x^{-n}J_{n+1}$

(c) $-x^nJ_n$ (d) $x^{-n}J_n$

19. The value of $J_{-1/2}(x)$ is :

(a) $\sqrt{\left(\frac{\pi x}{2}\right)}\cos x$ (b) $\sqrt{\left(\frac{2}{\pi x}\right)}\cos x$

(c) $\sqrt{\frac{2}{\pi x}}\sin x$ (d) $\sqrt{\frac{\pi x}{2}}\sin x$

20. The coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of $e^{\frac{x\left(2-\frac{1}{2}\right)}{2}}$ is :

(a) $P_n(x)$ (b) $J_n(x)$
(c) $(-1)^n J_n(x)$ (d) $(-1)^n P_n(x)$

21. If α, β are the roots of $J_n(x) = 0$ and $\alpha = \beta$ then $\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx$ is :
(a) 0 (b) $\frac{1}{2}[J_{n+1}(\alpha)]^2$
(c) $J_n(\alpha)J_n(\beta)$ (d) $\frac{J_n(\alpha)}{J_n(\beta)}$

22. J_0' is equal to :
(a) J_1 (b) $-J_1$ (c) $J_2 - J_1$ (d) $J_2 + 2J_1$

23. $x\,[J_{n-1} + J_{n+1}]$ is equal to :
(a) nJ_n (b) $-nJ_n$ (c) $2nJ_n$ (d) $-2nJ_n$

24. $J_0'' - \frac{J_0'}{x}$ is equal to :
(a) J_0 (b) J_2 (c) J_3 (d) J_2'

25. For $n \geq 1$, the result is :
(a) $|J_0(x)| \leq 1$ (b) $0 \leq J_0(x) \leq 1$
(c) $|J_0(x)| \leq \infty$ (d) $J_0(x) = 0$

26. $\frac{d}{dx}(x^n J_n)$ is equal to :
(a) $x^n J_n$ (b) $x^n J_{n+1}$ (c) $-x^n J_n$ (d) $x^n J_{n-1}$

27. $\int_0^x x^n J_{n-1}(x)\, dx$ is equal to :
(a) $x^{-n} J_n(x)$ (b) $x^n J_n(x)$
(c) $xJ_{n+1}(x)$ (d) $xJ_n(x)$

28. $\int_0^{\pi/2} \sqrt{\pi x}\, J_{1/2}(2x)\, dx$ is :
(a) 0 (b) $\frac{1}{\sqrt{\pi}}$ (c) $\frac{2}{\sqrt{\pi}}$ (d) 1

29. All the roots of $J_n(x) = 0$ are :
(a) Same
(b) Distinct
(c) More than one are repeated
(d) Repeated only at $x = 0$

30. The value of $J_n(x) - xJ_2(x)$ is :
(a) $xJ_1(x)$ (b) $J_1'(x)$
(c) $xJ_1'(x)$ (d) $-xJ_1'(x)$

31. $\lim_{x\to 0} \frac{J_n(x)}{x^n}$ for $n > -1$ is equal to :
(a) $\frac{2^n}{t(n+1)}$ (b) $\frac{1}{2^n t(n+1)}$
(c) $\frac{t\,(n+1)}{2^n}$ (d) $2^n t\,(n+1)$

32. $J_2 - J_0$ is equal to :
(a) J_0' (b) $2J_0'$ (c) $2J_0''$ (d) $-2J_0'$

33. For all $n \geq 1$:
(a) $|J_n(x)| \leq 1$ (b) $|J_n(x)| \leq \frac{1}{\sqrt{2}}$
(c) $0 \leq J_n(x) \leq 1$ (d) $J_n(x) = 0$

34. $J_0(x) - J_2(x)$ is equal to :
(a) $2J_1(x)$ (b) $J_1'(x)$
(c) $2J_1'(x)$ (d) $-J_1'(x)$

35. $\frac{d}{dx}[xJ_1(x)]$ is equal to :
(a) $xJ_0(x)$ (b) $J_1(x)$
(c) $xJ_1(x)$ (d) None of these

36. $J_n(x)$ is Bessel's polynomial then :
(a) $J_n(x) = (-1)^n J_{-n}(x)$ (b) $J_n(x) = (-1)^n J_n(x)$
(c) $J_n(x) = -J_{-n}(x)$ (d) None of these

37. The value of $J_2(-x)$ is equal to :
(a) $J_2(x)$ (b) $-J_2(x)$
(c) $J_{-2}(x)$ (d) None of these

38. $J_{-2}(x)$ is equal to :
(a) $J_2(-x)$ (b) $-J_2(x)$
(c) $J_2(x)$ (d) None of these

39. $\lim\limits_{x\to 0} \frac{J_1(x)}{x}$ is equal to :
(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 0

40. $\frac{d}{dx}[xJ_1(x)]$ is equal to :
(a) $J_0(x)$ (b) $J_1(x)$ (c) $-J_0(x)$ (d) $xJ_0(x)$

41. $x[J_1 + J_3]$ is equal to :
(a) J_2 (b) $2J_2$ (c) $3J_2$ (d) $4J_2$

If a and b are root of $J_n(x) = 0$ then

42. $\int_0^1 xJ_2(a^x)\, J_2(b^x)\, dx$ is equal to :
(a) 0 (b) a^2 (c) b^2 (d) $a^2 + b^2$

43. If $J_{n+1}(x) = \frac{2}{x}J_n(x) - J_0(x)$ then n is :
(a) 0 (b) 1
(c) 2 (d) None of these

44. $\int \frac{1}{x} J_2(x)\, dx$ is equal to :
(a) $-\frac{J_1(x)}{x} + c$ (b) $\frac{J_1(x)}{x} + c$
(c) $\frac{J_{-1}(x)}{x} + c$ (d) $J_0(x) + c$

45. The value of $\int x^2 J_1(x)\,dx$ is :

(a) $x^2 J_1(x) + c$
(b) $x^2 J_2(x) + c$
(c) $x^2 J_{-1}(x) + c$
(d) None of these

46. $\frac{d}{dx}[J_0(x)]$ is equal to :

(a) $J_0'(x)$
(b) $J_1'(x)$
(c) $-J_1(x)$
(d) $-J_2(x)$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(b)	**4.**	(c)	**5.**	(d)	**6.**	(b)	**7.**	(d)	**8.**	(c)	**9.**	(c)	**10.**	(c)
11.	(c)	**12.**	(a)	**13.**	(b)	**14.**	(c)	**15.**	(d)	**16.**	(a)	**17.**	(b)	**18.**	(a)	**19.**	(b)	**20.**	(b)
21.	(b)	**22.**	(b)	**23.**	(c)	**24.**	(b)	**25.**	(a)	**26.**	(d)	**27.**	(b)	**28.**	(d)	**29.**	(d)	**30.**	(c)
31.	(b)	**32.**	(c)	**33.**	(b)	**34.**	(c)	**35.**	(a)	**36.**	(a)	**37.**	(a)	**38.**	(c)	**39.**	(c)	**40.**	(d)
41.	(d)	**42.**	(a)	**43.**	(d)	**44.**	(a)	**45.**	(b)	**46.**	(c)								

HINTS AND SOLUTIONS

2. $$\because J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\tau(-n+r+1)}$$

If p is integer then $\tau(-p)$ is ∞ for $p \geq 0$ so terms in J_{-n} equal to zero till $-n+r+1 < 1$ *i.e.*, $r < n$. Hence

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\tau(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= \sum_{s=0}^{\infty} = \frac{(-1)^{n+s}}{n+s! + \tau\,(s+1)} \left(\frac{x}{2}\right)^{n+2s} \qquad \text{put } r = n+s$$

$$= (-1)^n \Sigma \frac{(-1)^s}{\tau\,(n+s+1)\,s!} \left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x)$$

7. $$\because J_n(x) \frac{x^n}{2^n(n+1)} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2n\,(2n+2)\,(2n+4)}\right]$$

Putting $n = \frac{1}{2}$ we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\tau\left(\frac{3}{2}\right)} \left[1 - \frac{x^2}{2.3} + \frac{x^4}{2.4.3.5}\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x.3}{3!} + \frac{x^5}{5!} \ldots.\right] = \sqrt{\frac{2}{\pi x}} \sin x$$

8. By Recurrence formula $-nJ_n(x) + xJ_{n-1}(x) = xJ_n'(x)$

9. $$\because \; J_n(x) = \frac{x^n}{2^n \tau\,(n+1)} \left[1 - \frac{x^2}{2\,(2n+2)} + \frac{x^4}{2.4(2n+2)\,(2n+4)}\right]$$

Putting $n = -\frac{1}{2}$ we get

$$J_{-1/2}(x) = \frac{\sqrt{x}}{\sqrt{2} + \left(\frac{1}{2}\right)} \left[1 - \frac{x^2}{2!} + \frac{x^4}{2.4.1.3} + \ldots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\right] = \sqrt{\frac{2}{\pi x}} \cos x$$

Similarly $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

So $J_{1/2}^2(x) + J_{-1/2}^2(x) = \frac{2}{\pi x}$

11. Since by generating function $J_n(x)$ is the coefficient of z^n in the expension of $\frac{e^{x(z-1/z)}}{2}$ in ascending or descending power of z for all positive integer n.

13. By orthogonal property of Bessel's functions

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}$$

where α, β are the roots of $J_n(x) = 0$.

18. By Recurrence formula $\frac{d}{dx}\{(x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$

21. By Orthogonal property of Bessel's function.

22. By Recurrence formula $xJ_n' = nJ_n - xJ_{n+1}$

Put $n = 0$ we have $xJ_0' = -xJ_1$

i.e. $J_0' = -J_1$

24. By Recurrence formula $xJ_n' = nJ_n - xJ_{n+1}$

putting $n = 1$, $xJ_1' = J_1 - xJ_2$

But we know that $J_0' = -J_1$

Differentiating it we get $J_0'' = -J_1'$ put it in above equation

we get $-xJ_0'' = -J_0' - xJ_2$ or $J_2 = J_0'' - \dfrac{J_0'}{x}$

25. Since we know that $J_0^2 + 2\,(J_1^2 + J_2^2 + \ldots + J_n^2 \ldots) = 1$

and since $J_1^2, J_2^2 \ldots$ are all non-negative so

$J_0^2(x) \le 1$ or $1J_0(x)^2 \le 1$ or $|J_0(x)| \le 1$

27. By Recurrence formula $x^n J_{n-1}(x) = \dfrac{d}{dx}[x^n J_n(x)]$

Integrating from 0 to x we get

$$\int_0^x x^n J_{n-1}(x)\,dx = [x^n J_n(x)]_0^x = x^n J_n(x)$$

28. $\because \quad J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$

$\Rightarrow \quad J_{1/2}(2x) = \sqrt{\dfrac{1}{\pi x}}\sin 2x$

$$\therefore \int_0^{\pi/2} \sqrt{\pi x}\,J_{1/2}(2x)\,dx = \int_0^{\pi/2} \sqrt{\pi x}\,.\sqrt{\frac{1}{\pi x}}\,.\sin 2x\,dx$$

$$= \int_0^{\pi/2} \sin 2x\,dx = \left[-\frac{\cos 2x}{2}\right]_0^{\pi/2} = 1$$

31. $J_n(x) = \dfrac{x^n}{2^n \tau(n+1)}\left[1 - \dfrac{x^2}{2.\,(2n+2)} + \ldots\right]$

$$\lim_{x\to 0} \frac{J_n(x)}{xn} = \lim_{x\to 0} \frac{1}{2^n(n+1)}\left[1 - \frac{x^2}{2\,(2n+2)} + \ldots\right]$$

$$= \frac{1}{2^n \tau(n+1)}$$

33. $\because J_0^2 + 2\,(J_1^2 + J_2^2 + \ldots J_n^2 + \ldots) = 1$

for $n \ge 1$, $2J_n^2(x) \le 1$

$\Rightarrow \quad 2\,|J_n(x)|^2 \le 1$

or $\quad |J_n(x)| \le \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad |J_n(x)| \le 2^{-1/2}$

34. By Recurrence formula $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$

Putting $n = 1$, $J_0 - J_2 = 2J_1'$

35. By Recurrence formula $\dfrac{d}{dx}(x^n J_n) = x^n J_{n-1}$

Put $n = 1$ we get $\dfrac{d}{dx}(xJ) = xJ_0$

37. $\because J_n(-x) = (-1)^n J_n(x)$ for all integers so putting $n = 2$ we get $J_2(-x) = J_2(x)$

38. $\because \; J_{-n}(x) = (-1)^n J_n(x)$ so putting $n = 2$ we get $J_{-2}(x) = J_2(x)$

39. $\because J_n(x) = \dfrac{x^n}{2^n \tau(n+1)}\left[1 - \dfrac{x^2}{2\,(2n+2)} + \ldots\right]$

Putting $n = 1$, $J_1(x) = \dfrac{x}{2\tau(2)}\left[1 - \dfrac{x^2}{8}\ldots\right]$

or $\quad \displaystyle\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2\tau(2)} = \frac{1}{2}$

40. By Recurrence formula

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

So $\quad$ putting $n = 1$, $\dfrac{d}{dx}(xJ_1) = -xJ_0$

41. By Recurrence formula

$x[J_{n-1} + J_{n+1}] = 2nJ_n$

Putting $n = 2$, $x[J_1 + J_3] = 4J_2$

42. By Orthogonal property

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = 0.$$

If α, β are the roots of $J_n(x) = 0$.

44. By Recurrence formula

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}(x)$$

Integrating both sides

$$\int x^{-n}J_{n+1}(x)\,dx = -[x^{-n}J_n(x)] + c$$

Putting $n = 1$, we get

$$\int x^{-1}J_2(x)\,dx = -x^{-1}J_1(x) + c = -\frac{J_1(x)}{x} + c$$

45. By Recurrence formula

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}$$

Integrating we get

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c$$

Putting $n = 2$ we get $\int x^2 J_1(x)\,dx = x^2 J_2(x) + c$

46. By Recurrence formula

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

Putting $\quad n = 0$ we get

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

* * *

Unit-III

CHAPTER

13

Series Solutions of Differential Equations

INTRODUCTION

1. Consider the linear differential equation of the second order

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \qquad ...(1)$$

Where P and Q are functions of x. Then there exists the following points.

(i) The point $x = 0$ is called ordinary point of (1) If $P(x)$ and$Q(x)$ do not become infinite in the neighbourhood of the origin and they can be expanded in the sum of power series.

(ii) The point $x = 0$ is called a singular point of the differential equation (1) if it is not an ordinary point. These are of two types.

(a) Regular singular points (b) Irregular singular points.

The origin is called regular singular point if both $xP(x)$ and $x^2Q(x)$ can be expanded in power series of x. Otherwise $x = 0$ is called an irregular singular points.

FROBENIUS METHOD

1. The method of finding a solution in a series for the differential equation (1) near a regular singular point or near an ordinary point is called frobenius method.

2. Solution near an ordinary point

Consider $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)\,y = 0$ ...(1)

Let its trial solution is $y = \sum_{n=0}^{\infty} C_n x^n$ ...(2)

Then $$\frac{dy}{dx} = \Sigma n C_n x^{n-1}$$

and $$\frac{d^2y}{dx^2} = \Sigma n\,(n-1)\,C_n x^{n-2}$$

If $P(x)$ and $Q(x)$ are not polynomials in x, then they can be expanded as

$$P(x) \sum_{n=0}^{\infty} P_n x^n \text{ and } Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Putting all these values in equation (1) and equating to zero the coefficient of various powers of x we determine all coefficients of (2). Putting these values in (2) we get the general solution of (1).

3. Solution near a regular singular point

Consider $\dfrac{d^2y}{dx^2} \neq P(x)\dfrac{dy}{dx} + Q(x)y = 0$ ...(1)

Let its trial solution is

$$y = x^m \sum_{n=0}^{\infty} C_n x^n \text{ with } C_0 \neq 0$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and substitute in (1) then it reduces to an identity in x. By equating to zero the coefficients of the lowest power of x in this identity we get quadratic equation in m called indicial equation. It determines m.

Now equating to zero the next coefficients of various powers of x we can determine all the coefficients $C_1, C_2, \ldots$ in terms of C_0.

4. Consider $y'' + P(x)\,y' + Q(x)\,y = 0$ and P and Q are not analytic ($P = \infty$ or $Q = \infty$) at $x = 9$ then $x = a$ is not ordinary point but a singular point. If $(x - a)\,P$ and $(x - a)^2 Q$ are infinite at $x = a$ then it is called irregular singular point. If $(x - a)\,P$ and $(x - a)^2 Q$ are not infinite at $x = a$ then it is called regular singular point.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. If $x = 0$ is not an ordinary point of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ then it is :
   (a) Regular point (b) Singular point
   (c) Essential point (d) None of these

2. If $xP(x)$ and $x^2Q(x)$ can be expanded in a power series of x in the *nbd* of $x = 0$ then $x = 0$ is called :
   (a) Regular point (b) Singular point
   (c) Ordinary point (d) Regular singular point

3. Singular points of a differential equation are :
   (a) Unique (b) Two types
   (c) Three types (d) May be infinite

4. If we put $y = \sum_{i=0}^{\infty} C_r x^{m+r}$ in $[xD^2 + (1+x)D + 2]\, y = 0$ then indicial equation is :
   (a) $C_0\,(m-1)^2$ (b) $C_0 m\,(m-1)$
   (c) $C_0 m^2$ (d) $C_0 m\,(m+1)$

5. The roots of indicial equation of $[x^2D^2 + xD + (x^2 - 1)]\, y = 0$ are :
   (a) 0, 1 (b) 0, –1 (c) 1, –1 (d) 0, 0

6. The indicial equation of $[(1-x^2)\, D^2 + 2xD + 1]\, y = 0$ is :
   (a) $C_0 m^2 = 0$ (b) $C_0 m\,(m-1) = 0$
   (c) $C_0\,(m-1)^2$ (d) $C_0 m\,(m+1)$

7. The indicial equation of $[xD^2 + D + x]\, y = 0$ is :
   (a) $C_0 m^2 = 0$ (b) $C_0(m-1)^2$
   (c) $C_0(m+1)^2$ (d) $C_0(m-1)\,(m-2)$

8. For the differential equation $(1-x^2)\, y'' - 2xy' + p\,(p+1)\, y = 0$ the point $x = 0$ is :
   (a) Regular (b) Singular
   (c) Ordinary (d) Essential

9. If $x = 1$ be the ordinary point of $y'' + (x-1)^2 y' - y\,(x-1)\, y = 0$ then the corresponding equation whose ordinary point $x = 0$ is :
   (a) $(D^2 + x_D^2 - 4x)\, y = 0$
   (b) $(D^2 - 4D)\, y = 0$
   (c) $(D^2 + xD^2 - 1)\, y = 0$
   (d) None of these

10. If $y = \Sigma C_r x^{m+r}$ be the solution of $9x(1-x)\, y'' - 12y' + 4y = 0$ then roots of indicial equations are :
   (a) 0, 1 (b) $0, \frac{3}{7}$ (c) 0, 3 (d) $0, \frac{7}{3}$

11. The indicial equation of $(D^2 - 1)\, y = x$ is :
   (a) $C_2 = \frac{C_0}{2}$ (b) $C_2 = \frac{C_1}{2}$
   (c) $m^2 = 0$ (d) $m\,(m-1) = 0$

12. The singular point of $x\,(x-1)\, y'' + xy' + x^2y = 0$ is :
   (a) 0 (b) 1
   (c) 2 (d) None of these

13. The singular point of $(x-1)y'' + xy' + 2y = 0$ is :
   (a) 3 (b) 2 (c) 1 (d) 0

14. For the equation $x\,(x-3)^2 y'' + 2\,(x-3)\, y' + xy = 0$ the regular singular points are :
   (a) 0 (b) 3 (c) 2 (d) 0, 3

15. For the solution of $x^2(x-n)\, y'' + (x-n)\, y' + y = 0$ at $x = 0$ choose y such that :
   (a) $y = \sum_{r=0}^{\infty} C_r x^{m+2r}$ (b) $\sum_{r=0}^{\infty} C_r x^{m-2r}$
   (c) $\sum_{r=0}^{\infty} C_r x^{m+r}$ (d) $\sum_{r=0}^{\infty} C_r x^{m+r-1}$

16. For the solution of differential equation $(x - x^2)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0$, y is equal to :
   (a) $\Sigma C_r x^r$ (b) $\Sigma C_r x^{m+r}$
   (c) $\Sigma C_r x^{m+2r}$ (d) None of these

17. The irregular singular point of $(x-1)\,(x-2)^3 y'' + (x-1)^2 y' + 3\,(x-1)\, y = 0$ is :
   (a) 0 (b) 1 (c) 2 (d) 3

18. For the equation $(D^2 + x^2)\, y = 0$, $x = 0$ is :
   (a) Ordinary point (b) Regular point
   (c) Singular point (d) None of these

19. The ordinary point of $y'' - \frac{x}{1-x^2} y' + \frac{y}{1-x^2} y = 0$ is :
   (a) 0 (b) 1
   (c) –1 (d) ± 1 both

20. At $x = 0$ the differential equation $y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0$ has :
(a) Regular singular point
(b) Irregular singular point
(c) Ordinary point
(d) Extreme point

21. At $x = 1$ the equation $x(x-1)y'' + xy' + x^2y = 0$, has :
(a) Singular point (b) Ordinary point
(c) Regular point (d) None of these

22. The singular point of $x^2(x-3)y'' + 3(x-3)y' + 5y = 0$ is :
(a) 1 (b) 0 (c) 3 (d) 5

23. For $(x-1)y'' + xy' + 3 = 0$, $x = 0$ is :
(a) Singular point (b) Regular singular point
(c) Irregular singular point
(d) Ordinary point

24. The regular singular point of $y'' + \frac{x-1}{x}y' + \frac{(x+2)}{x^2}y = 0$ is :
(a) 3 (b) 2 (c) 1 (d) 0

25. The roots of the indicial equation of $(xD^2 + D - 1)\,y = 0$ are :
(a) 0, 0 (b) 0, 1 (c) 0, – 1 (d) 1, – 1

26. The indicial equation of $[x^2D^2 + xD + (x^2 - 4)]\,y = 0$ is :
(a) $C_0(m-2)^2$ (b) $C_0(m+2)^2$
(c) $C_0(m+2)(m-2)$ (d) $C_0m\,(m-2)$

27. The roots of the indicial equation of $(x^2D^2 + 5xD + x^2)\,y = 0$ are :
(a) 0, 1 (b) 0, 2 (c) 0, 4 (d) 0, –4

28. If $y = \sum\limits_{r=0}^{\infty} C_r x^{m+r}$ be the solution of $[(x - x^2)\,D^2 + (1 - 5x)\,D - 4]\,y = 0$ then its indicial equation is :
(a) $C_0m\,(m-1) = 0$ (b) $C_0(m+1)(m-1) = 0$
(c) $C_0m\,(m-2) = 0$ (d) C_0m^2

29. If $y = \sum\limits_{r=0}^{\infty} C_r x^{m+2r}$ be the solutions of $(2x + x^3)\,D^2y - Dy - 6xy = 0$ then its indicial equation is :
(a) $C_0m\,(m-2) = 0$ (b) $C_0m\,(2m-3) = 0$
(c) $C_0m\,(2m-1) = 0$ (d) None of these

30. The indicial equation of $2x^2D^2 - xD + (1 - x^2)\,y = x^2$ is :
(a) $C_0(2m-1)(m-1) = 0$
(b) $C_0(m-1)(m-2) = 0$
(c) C_0m^2
(d) $C_0m\,(m-1) = 0$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(d)	**3.**	(b)	**4.**	(c)	**5.**	(c)	**6.**	(b)	**7.**	(a)	**8.**	(c)	**9.**	(a)	**10.**	(d)
11.	(a)	**12.**	(b)	**13.**	(c)	**14.**	(d)	**15.**	(c)	**16.**	(a)	**17.**	(c)	**18.**	(c)	**19.**	(a)	**20.**	(b)
21.	(a)	**22.**	(b)	**23.**	(d)	**24.**	(d)	**25.**	(a)	**26.**	(c)	**27.**	(d)	**28.**	(d)	**29.**	(b)	**30.**	(a)

HINTS AND SOLUTIONS

4. Given $y = \Sigma C_r x^{m+r}$ be the solution of

$$x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0 \qquad \ldots(1)$$

so $$\frac{dy}{dx} = \Sigma C_r(m+r)x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \Sigma C_r(m+r)(m+r-1)\,x^{m+r-2}$$

put in (1) we get

$\Sigma C_r[(m+r+2)\,x^{m+r} + (m+r)^2x^{m+r-1}] = 0$

Equating to zero the coefficient of the lowest power of x *i.e.* x^{m-1} we get indicial equation (1) $C_0m^2 = 0$

5. Put $y = \Sigma C_r x^{m+2r}$ be the solution of given equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)\,y = 0$$

then $\frac{dy}{dx} = \Sigma C_r(m+2r)\,x^{m+2r-1}$,

$$\frac{d^2y}{dx^2} = \Sigma C_r(m+2r)(m+2r-1)\,x^{m+2r-2}$$

put these in given equation we get

$$\Sigma C_r[(m+2r+1)(m+2r-1)\,x^{m+2r} + x^{m+2r+2}] = 0$$

Equating the coefficient of lowest power is x^m we get

$$C_0(m+1)(m-1) = 0$$

$$\Rightarrow \qquad m = 1, -1$$

8. Given equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p\,(p+1)\,y = 0$$

i.e. $$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{p\,(p+1)}{(1-x^2)}y = 0$$

Here, $$P = -\frac{2x}{1-x^2},\ Q = \frac{p\,(p+1)}{1-x^2}$$

Since at $x = 0$

neither P nor $Q \to \infty$ so $x = 0$ is an ordinary point

9. Given that $x = 1$ be the ordinary point of

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} = -4\,(x-1)\,y = 0$$

put $x = t+1$, $\frac{dy}{dx} = \frac{dy}{dt} - \frac{dt}{dx} = \frac{dy}{dt}$

So, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

So given equation becomes

$$\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} - 4ty = 0$$

and its ordinary point is $t = 0$

11. Let $y = \Sigma C_r x^r$ be the solution of $D^2y - y = x$

So $\frac{dy}{dx} = \Sigma C_r r x^{r-1}$ and $\frac{d^2y}{dx^2} = \Sigma C_r r(r-1)\,x^{r-2}$

Put in given equation we get

$$\Sigma C_r[r\,(r-1)\,x^{r-2} - x^r] - x = 0$$

Equating to zero the coefficient of x^0 we get

$$2A_2(2-1) - A_0 = 0$$

$$\Rightarrow \qquad A_2 = \frac{A_0}{2}$$

12. Given differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x-1}\frac{dy}{dx} + \frac{x}{x-1}y = 0$$

Here, $$P = \frac{1}{x-1},\ Q = \frac{x}{x-1}$$

So $(x-1)\,P$ and $(x-1)^2Q$

i.e. 1 and $x\,(x-1)$ both are not infinite at $x = 1$ so $x = 1$ is regular point.

14. Given equation is

$$\frac{d^2y}{dx^2} + \frac{2}{x\,(x-3)}\frac{dy}{dx} + \frac{1}{(x-3)^2}y = 0$$

Here $$P = \frac{2}{x\,(x-3)} \text{ and } Q = \frac{1}{(x-3)^2}$$

It is obvious that xP and x^2Q both are not infinite at $x = 0$ so $x = 0$ is regular singular point.

Also $(x-3)$ and $(x-3)^2Q$ both are not infinite at $x = 3$ so $x = 3$ is regular singular point.

16. Given equation is

$$(x-x^2)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0$$

Put x^m in given equation we get

$$(x-x^2)m\,(m-1)\,x^{m-2} + (1-5x)\,mx^{m-1} - 4x^m$$

or $$m^2x^{m-1} - (m^2+4m+4)\,x^m$$

Here the common difference of the power is so its solution is $y = \sum_{r=0}^{\infty} C_r x^{m+r}$

17. Given that $$\frac{d^2y}{dx^2} + \frac{x-1}{(x-2)^3}\frac{dy}{dx} + \frac{3}{(x-2)^3}y = 0$$

Here, $$P = \frac{x-1}{(x-2)^3},\ Q = \frac{3}{(x-2)^3}$$

and $(x-2)\,P$ and $(x-2)^2Q$ are infinite at $x = 2$ so, $x = 2$ is an irregular singular point.

20. Given differential equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x^3}y = 0$$

Here, $$P = \frac{1}{x^2},\ Q = \frac{1}{x^3}$$

and xP and x^2Q are infinite at $x = 0$ so $x = 0$ is a regular singular point.

22. Given that $x^2(x-3)\,y'' + 3\,(x-3)\,y' + 5y = 0$

or $$\frac{d^2y}{dx^2} + \frac{3}{x^2}\frac{dy}{dx} + \frac{5}{x^2(x-3)}y = 0$$

Here, $$P = \frac{3}{x^2} \text{ and } Q = \frac{5}{x^2(x-3)}$$

Also P and Q are not analytic at $x = 0$

i.e. $P = \infty$, $Q = \infty$

so $x = 0$ is a singular point.

24. Given differential equation is

$$\frac{d^2y}{dx^2} + \frac{x-1}{x}\frac{dy}{dx} + \frac{x+2}{x^2}y = 0$$

Here, $P = \frac{x-1}{x}, Q = \frac{x+2}{x^2}$

Also xP and x^2Q are not ∞ at $x = 0$ so $x = 0$ is a regular singular point.

26. Put $y = \Sigma C_r x^{m+r}$ in $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\,y = 0$

We get $\Sigma c_r(m + r + 2)\,(m + r - 2)\,x^{m+r}$

$$+\Sigma C_r x^{m+r+2} = 0$$

So equating the lowest degree term *i.e.* x^m we get $C_0(m + 2)\,(m - 2) = 0$ which is indicial equation.

30. Put $y = \sum_{r=0}^{\infty} C_r x^{m+2r}$ in given equation we have

$$\frac{dy}{dx} = \Sigma C_r(m + 2r)\,x^{m+2r-1},$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} C_r(m + 2r)(m + 2r - 1)\,x^{m+2r-2}$$

We get $\Sigma C_r[-x^{m+2r+2} + (2m + 4r - 1)$

$$(m + 2r - 1)\,x^{m+2r}] = 0$$

So equating to zero the coefficient of lowest term *i.e.* x^m is

$$C_0(2m - 1)\,(m - 1) = 0$$

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Unit-III

CHAPTER

14

The Laplace Transform

LAPLACE TRANSFORM

1. The Laplace transform reduces the problem of solving a differential equation to an algebraic problem.

   If the knernel $k(p, t)$ is defined by
   $$k\,(p,\,t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-pt} & \text{for } t \geq 0 \end{cases}$$
   Then$f(p) = L\,\{F(t)\} = \bar{F}(p)$
   $$= \int_0^\infty \bar{e}^{pt} F(t)\,dt \qquad ...(1)$$
   is called the Laplace transform of the function $F(t)$. It exists if the integral (1) converger for some values of p, otherwise it does not exist.

2. The Laplace transform is a linear transformation *i.e.*

   $L\,\{a_1F_1(t) + a_2F_2(t)\} = a_1L\,\{F_1(t)\} + a_2L\,\{F_2(t)\}$

   Where a_1, a_2 are constants.

3. A function $F(t)$ is said to be of exponential order α as $t \to \infty$ if there exists a positive constant μ, a number α and a finite number to such that $|F(t)| < \mu e^{\alpha t}$ or

   $|e^{-\alpha t}F(t)| < \mu$ for all $t \geq t_0$

   If a function $F(t)$ is of exponential order α then it is also of β for $\beta > \alpha$.

4. If $F(t)$ is a function which is piecewise continuous on every finite internal in the range $t \geq 0$ and satisfies $|F(t)| \leq \mu e^{at}$ for all $t \geq 0$ and for some constant a and μ then the Laplace transform of $F(t)$ exists for at $p > a$.

5. A function which is piecewise continuous on every finite interval in the range $t \geq 0$ and is of exponential order as $t \to \infty$ is known as a function of class A.

6. Laplace transforms of some elementary functions are

   (i) $L\,\{1\} = \dfrac{1}{p}$, $p > 0$

   (ii) $L\{t^n\} = \dfrac{\tau\,(n+1)}{p^{n+1}}$, $p > 0$

   (iii) $L\,\{t^n\} = \dfrac{n\,!}{p^n + 1}$, $p > 0$ and n is a positive integer.

   (iv) $L\{e^{at}\} = \dfrac{1}{p-a}$, $p > a$

   (v) $L\{\sin at\} = \dfrac{a}{p^2 + a^2}$, $p > 0$

   (vi) $L\,\{\cos at\} = \dfrac{p}{p^2 + a^2}$, $p > 0$

   (vii) $L\,\{\sinh at\} = \dfrac{a}{p^2 - a^2}$, $p > |a|$

   (viii) $L\,\{\cosh at\} = \dfrac{p}{p^2 - a^2}$, $p > |a|$

PROPERTIES OF LAPLACE TANSFORMS

1. If $L\,\{F(t)\} = f(p)$ when $p > \alpha$

   then $L\,\{e^{at}F(t)\} = f(p-a)$, $p > \alpha + a$

   This is called first translation or shifting theorem.

2. If $L\,\{F(t)\} = f(p)$ and
   $$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$
   then $L\,\{G(t)\} = e^{-ap} f(p)$

   This is called second translation or shifting theorem.

3. If $L\,\{F(t)\} = f(p)$, then
   $$L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$$
   This is called change of scale property.

4. Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order a as or $t \to \infty$ and if $F'(t)$ is of class A, then Laplace transform of the derivative $F'(t)$ exists when $p > a$ and $L\{F'(t)\} = pL\{F(t)\} - F(0)$

   Similarly

   $L\{F''(t)\} = p^2 L\{F(t)\} - pF(0) - F'(0)$

   Continuing in this manners we have

   $$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \ldots - F^{n-1}(0)$$

5. Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then

   $$\lim_{t \to 0} F(t) = \lim_{p \to \infty} pL\{F(t)\}$$

   This is called initial-value theorem.

6. Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim\limits_{t \to \infty} F(t) = \lim\limits_{p \to 0} pL\{F(t)\}$

   This is called final-value theorem.

7. If $F(t)$ is piecewise continuous and satisfies $|F(t)|\,\Sigma\mu e^{at}$ for all $t \geq 0$ for some constants a and μ, then

   $$L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{p} L\{F(t)\},\ (p > 0,\ p > a)$$

8. If $F(t)$ is a function of class A and if $LF(t) = f(p)$, then

   $$L\{tF(t)\} = -f'(p)$$

   Similarly $L\{t^n F(t)\} = (-1)^n \dfrac{dn}{dp^n} f(p)$

   For all $n = 1, 2, 3, \ldots$

9. If $L\{f(t)\} = f(p)$, then $L\left\{\dfrac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$

   provided $\lim\limits_{t \to 0}\left\{\dfrac{F(t)}{t}\right\}$ exists.

10. Let $L\{F(t)\} = f(p)$ i.e. $\int_0^\infty e^{-pt}F(t)\,dt = f(p)$

    taking limit as $p \to 0$, we have

    $$\int_0^\infty F(t)\,dt = f(0)$$

11. Let $F(t)$ be a periodic function with period $T > 0$, $F(u + T) = F(u)$, $F(u + 2T) = F(u)$ etc.

    then $\quad L\{F(t)\} = \dfrac{\int_0^T e^{-pt}F(t)\,dt}{1 - e^{-pt}}$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The Laplace transform of t is :

   (a) $\dfrac{1}{p^3}$ (b) $\dfrac{1}{p^2}$ (c) $\dfrac{2}{p^2}$ (d) 1

2. $L\{\cosh at\}$ is equal to :

   (a) $\dfrac{1}{p^2 - a^2}$ (b) $\dfrac{p}{p^2 + a^2}$

   (c) $\dfrac{p}{p^2 - a^2}$ (d) $\dfrac{1}{p^2 + a^2}$

3. If $L\{F(t)\} = f(p)$ then $L\{F(at)\}$ is :

   (a) $f\left(\dfrac{p}{a}\right)$ (b) $af\left(\dfrac{p}{a}\right)$

   (c) $\dfrac{1}{a} f\left(\dfrac{p}{a}\right)$ (d) None of these

4. $L\{(5t - 2)\}$ is :

   (a) $\dfrac{5 - 2p}{p^2}$ (b) $\dfrac{5p - 2}{p^2}$

   (c) $\dfrac{5 - 2p^2}{p}$ (d) None of these

5. $L\{t^5 e^{3t}\}$ is :

   (a) $\dfrac{5}{(p-3)^6}$ (b) $\dfrac{60}{(p-3)^6}$

   (c) $\dfrac{120}{(p-3)^6}$ (d) $\dfrac{15}{(p-3)^6}$

6. $L\left\{\dfrac{\cos at}{t}\right\}$ is equal to :

   (a) $\cot^{-1}\left(\dfrac{p}{a}\right)$ (b) $\tan^{-1}\left(\dfrac{p}{a}\right)$

   (c) $\sin^{-1}\left(\dfrac{p}{a}\right)$ (d) Does not exist

7. The function e^{t^2} as $t \to \infty$ is of exponential order :

(a) 1 (b) 2 (c) 3 (d) Not exist

8. The value of $L\,\{\sin t \cos t\}$ is :

(a) $\dfrac{1}{p^2+1}$ (b) $\dfrac{2}{p^2+2}$ (c) $\dfrac{1}{p^2+4}$ (d) $\dfrac{2}{p^3+4}$

9. If $F(t) \Rightarrow \begin{cases} e^t & r < t \le 1 \\ 0 & t > 1 \end{cases}$ then $L\{F(t)\}$ is :

(a) $\dfrac{1-e^{-p}}{p-1}, p \ne 1$ (b) $\dfrac{1-e^{-p+1}}{p-1}, p \ne 1$

(c) $\dfrac{1+e^{-p+1}}{p-1}, p \ne 1$ (d) None of these

10. If $F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0 & t > \pi \end{cases}$ then $L\{F(t)\}$ is :

(a) $\dfrac{e^{\pi p}}{p^2+1}$ (b) $\dfrac{e^{\pi p}+1}{p^2+1}$

(c) $\dfrac{e^{-\pi p}+1}{p^2+1}$ (d) None of these

11. The value of $L\left\{\dfrac{1}{\sqrt{\pi t}}\right\}$ is :

(a) $\dfrac{1}{p}, p > 0$ (b) $\dfrac{1}{\sqrt{p}}, p > 0$

(c) $\dfrac{1}{\sqrt{\pi p}}, p > 0$ (d) $\dfrac{p}{\sqrt{\pi}}, p > 0$

12. The exponential order of t^2 is :

(a) 1 (b) 2 (c) 3 (d) Not exist

13. $L\,\{e^t \sin^2 t\}$ is equal to :

(a) $\dfrac{2}{(p-1)\,(p^2+5)}$ (b) $\dfrac{2}{(p+1)\,(p^2-2p+5)}$

(c) $\dfrac{2}{(p+1)\,(p^2-5)}$ (d) $\dfrac{2}{(p-1)\,(p^2-2p+5)}$

14. If $F(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & t < \dfrac{2\pi}{3} \end{cases}$

Then $L\,\{F(t)\}$ is :

(a) $\dfrac{e^{-\pi \frac{p}{3}}}{p\,(p^2+1)}$ (b) $\dfrac{e^{-2\pi \frac{p}{3}}}{p^2+1}$

(c) $\dfrac{pe^{-2\pi \frac{p}{3}}}{p^2+1}$ (d) $\dfrac{pe^{2\pi \frac{p}{3}}}{p^2+1}$

15. $L\,\{t^n e^{at}\}$ is :

(a) $\dfrac{n!}{p^{n+1}}$ (b) $\dfrac{n-1!}{(p+a)^{n+1}}$

(c) $\dfrac{n!}{(p+a)^{n+1}}$ (d) $\dfrac{n!}{(p-a)^{n+1}}$

16. The value of $L\,\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$ is :

(a) $\dfrac{p-6}{p^2+4p+40}$ (b) $\dfrac{3p-24}{p^2+4p+40}$

(c) $\dfrac{3p-9}{p^2+4p+40}$ (d) None of these

17. If $L\,\{F(t)\} = f(p)$ then $L\{F(t) \cos at\}$ is :

(a) $\dfrac{f(p-ia)-f(p+ia)}{2}$ (b) $\dfrac{f(p-a)+f(p+a)}{2}$

(c) $\dfrac{f(p+ia)+f(p-ia)}{2}$ (d) $\dfrac{f(p-a)-f(p+a)}{2}$

18. $L\{\sin 2t \sin 3t\}$ is : ***[Kanpur 2012]***

(a) $\dfrac{6p}{(p^2+1)\,(p^2+25)}$ (b) $\dfrac{12p}{(p^2+1)\,(p^2+25)}$

(c) $\dfrac{p}{(p^2+1)\,(p^2+25)}$ (d) None of these

19. If $L\,\{F(t)\} = f(p)$ then $L\,\{e^{-t}F(3t)\}$ is :

(a) $\dfrac{e^{\frac{-3}{(p+1)}}}{(p+1)}$ (b) $\dfrac{e^{-\frac{3}{p}}}{(p+1)}$

(c) $\dfrac{e^{-\frac{3}{(p+1)}}}{p}$ (d) $\dfrac{e^{\frac{3}{p+1}}}{(p+1)}$

20. The value of $L\,\{t \sin at\}$ is :

(a) $\dfrac{ap}{(p^2+a^2)}$ (b) $\dfrac{2ap}{(p^2+a^2)}$

(c) $\dfrac{2ap}{(p^2+a^2)^2}$ (d) None of these

21. If $L\left\{\dfrac{\sin t}{t}\right\} = \tan^{-1}\dfrac{1}{p}$ then $L\left\{\dfrac{\sin at}{t}\right\}$ is :

(a) $\dfrac{1}{a}\tan^{-1}\left(\dfrac{a}{p}\right)$ (b) $\dfrac{1}{a}\tan^{-1}\left(\dfrac{p}{a}\right)$

(c) $a\tan^{-1}\left(\dfrac{a}{p}\right)$ (d) $\tan^{-1}\left(\dfrac{a}{p}\right)$

22. $L\,\{t^n e^{at}\}$ is equal to :

(a) $\dfrac{n!}{p^{n+1}}$ (b) $\dfrac{n!}{(p+a)^{n+1}}$

(c) $\dfrac{n!}{(p-a)^{n+1}}$ (d) $\dfrac{n}{(p+a)^{n+1}}$

23. The value of $L\left\{\dfrac{\sin ht}{t}\right\}$ is :

(a) $\log\dfrac{p+1}{p-1}$ (b) $\log\dfrac{p-1}{p+1}$

(c) $\frac{1}{2}\log\frac{p-1}{p+1}$ (d) None of these

24. If $L\{F(t)\} = f(p)$ and integral is converges then $\int_0^\infty \frac{F(t)}{t}\,dt$ is :

(a) $\int_0^\infty x f(x)\,dx$ (b) $\int_0^\infty \frac{f(x)}{x}\,dx$

(c) $\int_0^\infty f(x)\,dx$ (d) Not exist

25. The value of $\int_0^\infty te^{-t}\sin t\,dt$ is :

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

26. The value of $\int_0^\infty \frac{\sin t}{t}\,dt$ is :

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{p}{\pi}$ (d) 0

27. The value of $\int_0^\infty te^{-3t}\sin t\,dt$ is :

(a) $\frac{1}{25}$ (b) $\frac{2}{25}$ (c) $\frac{3}{25}$ (d) $\frac{4}{25}$

28. If $L\{J_0(t)\} = \frac{1}{\sqrt{1+p^2}}$ then $\int_0^\infty J_0(t)\,dt$ is :

(a) π (b) ∞ (c) 0 (d) 1

29. If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{p^{3/2}}$ then $L\left\{\frac{1}{\sqrt{\pi t}}\right\}$ is :

(a) $\frac{1}{\sqrt{p}}$ (b) $\frac{1}{p}$ (c) $\frac{1}{p\sqrt{p}}$ (d) $\sqrt{\frac{2}{p}}$

30. If $F(t) = \begin{cases} \sin t, & r < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ and $F(t)$ has a period of 2π then $L\{F(t)\}$ is :

(a) $\frac{1}{(p^2-1)(1-e^{-p\pi})}$ (b) $\frac{1}{(p^2+1)\,e^{\pi p}}$

(c) $\frac{1}{(p^3+1)(1+e^{-p\pi})}$ (d) $\frac{1}{(p^2+1)\,e^{-\pi p}}$

31. The value of $L\left\{\frac{1-e^{-2t}}{t}\right\}$ is :

(a) $\log\left(1+\frac{2}{p}\right)$ (b) $\frac{\log 2}{p}$

(c) $\log\left(1-\frac{2}{p}\right)$ (d) $\log\left(1+\frac{2}{p^2}\right)$

32. If $F(t) = \begin{cases} 0, 0 < t < 1 \\ t, 1 < t < 2 \\ 0, 2 < t \end{cases}$ then, $L\{F(t)\}$ is under $\lim\limits_{p\to 1}$ is :

(a) $-\frac{3}{e}+\frac{2}{e^2}$ (b) $-\frac{3}{e^2}+\frac{2}{e}$

(c) $\frac{3}{e^2}-\frac{2}{e}$ (d) $\frac{3}{e}+\frac{2}{e^2}$

33. The value of $L\{t^2e^t\}$ is :

(a) $\frac{2}{p^3}$ (b) $\frac{1}{(p-1)^3}$

(c) $\frac{2}{(p-1)^3}$ (d) $\frac{p}{(p-1)^3}$

34. $\int_0^\infty \frac{e^{-x}\sin x}{x}\,dx$ is equal to :

(a) $\frac{\pi}{2}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

35. If $L\{J_1(t)\} = 1-\frac{p}{\sqrt{p^2+1}}$ then $L\{t\,J_1(t)\}$ is :

(a) $\frac{1}{p^2+1}$ (b) $\frac{1}{\sqrt{p^2+1}}$

(c) $\frac{1}{(p^2+1)^{3/2}}$ (d) $\frac{1}{(p^2+1)^{5/2}}$

36. If $F(t) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$ then $L\{F(t)\}$ is :

(a) $\frac{e^{ap}}{p}$ (b) $\frac{e^{-ap}}{p}$ (c) $\frac{e^{-a/p}}{p}$ (d) $\frac{e^{-p/a}}{p}$

37. If $F(t) = \begin{cases} \sin\left(t-\frac{\pi}{3}\right) & t > \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$ then, $L\{F(t)\}$ is :

(a) $\frac{e^{\pi p/3}}{p+1}$ (b) $\frac{e^{\pi p/3}}{p^2+1}$

(c) $\frac{e^{-\pi p/3}}{p^2+1}$ (d) None of these

38. The value of $L\{t^ne^{nt}\}$ is :

(a) $\frac{n!}{p^{n+1}}$ (b) $\frac{n!}{(p-n)^{n+1}}$

(c) $\frac{n!}{(p+n)^{n+1}}$ (d) $\frac{n!}{(p-n)^n}$

39. $L\{t\sin at\}$ is equal to :

(a) $\frac{2ap}{(p^2+a^2)}$ (b) $\frac{2p}{(p^2+a^2)^2}$

(c) $\frac{2ap}{(p^2+a^2)^2}$ (d) None of these

40. $L\left\{\int_0^t \frac{\sin x}{x}dx\right\}$ is equal to :

(a) $\frac{\cot^{-1}p}{p}$ (b) $\frac{\tan^{-1}p}{p}$

(c) $\frac{\sin p}{p}$ (d) $\frac{\sin^{-1}p}{p}$

41. The value of $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$ is :

(a) $\frac{3}{p}$ (b) $\frac{3}{p^2}$ (c) $\log 3$ (d) $\frac{\log 3}{p^2}$

42. $\int_0^\infty te^{-3t}\cos xt\,dt$ is equal to :

(a) $\frac{7}{625}$ (b) $\frac{7}{225}$ (c) $\frac{1}{625}$ (d) $\frac{1}{225}$

43. If $L\ \{erf\sqrt{t}\} = \frac{1}{p\sqrt{p+1}}$ then $L\{e^{3t}erf\sqrt{t}\}$ is equal to :

(a) $\frac{1}{(p+3)\sqrt{p-2}}$ (b) $\frac{1}{(p-3)\sqrt{p+2}}$

(c) $\frac{1}{(p-3)\ \sqrt{p-2}}$ (d) $\frac{1}{(p-3)\sqrt{p}}$

44. If $L\{F(t)\} = f(p)$ then $L\{F(at)\}$ is :

(a) $\frac{1}{a}f(ap)$ (b) $af\left(\frac{p}{a}\right)$

(c) $\frac{1}{a}f\left(\frac{p}{a}\right)$ (d) $\frac{1}{a}f\left(\frac{a}{p}\right)$

45. $L\left[\frac{e^{-at}t^{n-1}}{n!-1!}\right]$ is equal to :

(a) $\frac{n!}{(p-a)^n}$ (b) $\frac{1}{(p+a)^n}$

(c) $\frac{1}{(p-a)^n}$ (d) $\frac{n!}{(p+a)^n}$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(c)	**4.**	(a)	**5.**	(c)	**6.**	(d)	**7.**	(d)	**8.**	(c)	**9.**	(b)	**10.**	(c)
11.	(b)	**12.**	(c)	**13.**	(d)	**14.**	(c)	**15.**	(d)	**16.**	(b)	**17.**	(a)	**18.**	(b)	**19.**	(a)	**20.**	(c)
21.	(d)	**22.**	(c)	**23.**	(d)	**24.**	(c)	**25.**	(c)	**26.**	(b)	**27.**	(c)	**28.**	(d)	**29.**	(a)	**30.**	(a)
31.	(a)	**32.**	(b)	**33.**	(c)	**34.**	(d)	**35.**	(c)	**36.**	(b)	**37.**	(c)	**38.**	(b)	**39.**	(c)	**40.**	(a)
41.	(c)	**42.**	(a)	**43.**	(c)	**44.**	(c)	**45.**	(b)										

HINTS AND SOLUTIONS

5. $L\{t^5\} = \frac{5!}{p^6} = \frac{120}{p^6}$

$\therefore$ By first shfting theorem.

$$L\{e^{3t}t^5\} = \frac{120}{(p-3)^6}$$

6. Since $\lim\limits_{t\to 0}\frac{\cos at}{t}$ does not exist so $L\left\{\frac{\cos at}{t}\right\}$ not exist.

7. $\lim\limits_{t\to\infty}\ \{e^{-at}e^{t^2}\} = \lim\limits_{t\to\infty} e^{t(t-a)} = \infty$ for all values of a.

Hence for any value of a, we can not find a number μ such that $e^{t^2} < \mu e^{at}$ so e^{t^2} is not of exponential order as $t \to \infty$.

9. $F(t) = \begin{cases} e^t & 0 < t \le 1 \\ 0 & t > 1 \end{cases}$ so $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

$$= \int_0^1 e^{-pt}e^t dt + \int_0^\infty e^{-pt}0\,dt = \left(\frac{e^{-(p-1)t}}{p-1}\right)_0^1$$

$$= \frac{1-e^{-(p-1)}}{p-1}, p \ne 1$$

12. $\lim\limits_{t\to\infty}\{e^{-at}F(t)\} = \lim\limits_{t\to\infty}\left(\frac{t^2}{e^{at}}\right) = \lim\limits_{t\to\infty}\left(\frac{2t}{ae^{at}}\right)$

$$= \lim_{t\to\infty}\frac{2}{a^2e^{at}} = 0 \text{ if } a > 0$$

so $F(t) = t^2$ is of exponential order.

$\because$ $|t^2| = t^2 < e^{3t}\forall t > 0$

So t^2 is of exponential order 3.

14. Let $\phi(t) = \cos t$ then $F(t) = \begin{cases} \phi\left(t-\frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

So $L\{\phi(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = f(p)$

From second shifting theorem we have

$$L\{F(t)\} = e^{-\left(\frac{2\pi}{3}\right)p} . f(p) = e^{-2\pi\frac{p}{3}} . \frac{p}{p^2+1}$$

16. ∵ $L\{3\cos 6t - 5\sin 6t\} = 3\frac{p}{p^2+6^2} - 5.\frac{6}{p^2+6^2}$

$= \frac{3p-30}{p^2+36} = f(p)$ so by first shifting theorem.

We have $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(pt^2)$

$$= \frac{3p-24}{p^2+4p+40}$$

17. ∵ $L\{F(t)\} = f(p)$ then by first shifting theorem

$$L\{f(t)\cos at\} = L\left\{F(t)\frac{1}{2}(e^{iat}-e^{-iat})\right\}$$

$$=\frac{1}{2}[L\{f(t)e^{-iat}\} - L\{F(t)e^{-iat}\}]$$

$$=\frac{1}{2}[f(p-ia)-f(p+ia)]$$

18. $F(t) = \sin 2t\sin 3t = \frac{1}{2}(\cos t - \cos 5t)$

So, $L\{F(t)\} = \frac{1}{2}[L(\cos t) - L\{\cos 5t\}]$

$$=\frac{1}{2}\left[\frac{p}{p^2+1} - \frac{p}{p^2+25}\right]$$

$$=\frac{p}{2}\left\{\frac{p^2+25-p^2+1}{(p^2+1)(p^2+25)}\right\} = \frac{12p}{(p^2+1)(p^2+25)}$$

20. $L\{\sin at\} = \frac{a}{p^2+a^2}$

∴ $L\{t\sin at\} = -\frac{d}{dp}\left\{\frac{a}{p^2+a^2}\right\} = \frac{2ap}{(p^2+a^2)^2}$

21. Given $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right) = L\left\{\frac{\sin at}{at}\right\}$

$$=\frac{1}{a}\tan^{-1}\left(\frac{a}{p}\right)$$

So $L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin at}{at}\right\}\frac{a}{a}\tan^{-1}\left(\frac{a}{p}\right) = \tan^{-1}\left(\frac{a}{p}\right)$

23. Here, $F(t) = \sin ht$

so, $\lim_{t\to 0}\frac{F(t)}{t} = \lim\frac{\sin ht}{t} = 1$

since $L\{\sin ht\} = \frac{1}{p^2-1} = f(p)$

so $L\left\{\frac{\sin ht}{t}\right\} = \int_p^\infty f(x)dx = \int_p^\infty \frac{dx}{x^2-1}$

$$=\left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_p^\infty = \left[\frac{1}{2}\log\frac{1-\frac{1}{x}}{1+\frac{1}{x}}\right]_p^\infty = \frac{1}{2}\log\frac{p+1}{p-1}$$

25. Let $F(t) = t\sin t$

$L\{F(t)\} = L\{t\sin t\} = -\frac{d}{p}L\{\sin t\} = -\frac{d}{dp}\frac{1}{p^2+1}$

$$=\frac{2p}{(p^2+1)^2}$$

so, $\int_0^\infty e^{-pt}t\sin t\,dt = \frac{2p}{(p^2+1)^2}$

letting $p \to 1$ we get

$$\int_0^\infty e^{-t}\,t\sin t\,dt = \frac{2}{4} = \frac{1}{2}$$

26. Let $F(t) = \sin t$ $\Rightarrow L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1}$

$$L\left[\frac{\sin t}{t}\right] = \int_0^\infty e^{-pt}\frac{\sin t}{t}dt = \int_p^\infty f(x)\,dx$$

$$=\int_p^\infty\frac{1}{x^2+1}dx = [\tan^{-1}C]_p^\infty = \frac{\pi}{2} - \tan^{-1}p$$

put $p \to 0$ we get

$$\int_0^\infty\frac{\sin t}{t}dt = \frac{\pi}{2}$$

29. Given $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{p^{3/2}}$

Let $F(t) = 2\sqrt{\frac{t}{\pi}}$ so $F'(t) = \frac{1}{\sqrt{\pi t}}$

∴ $L\{F'(t)\} = pL\{F(t)\} - F(0)$

or $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = pL\left\{2\sqrt{\frac{t}{\pi}}-0\right\} = \frac{p}{p^{3/2}} = \frac{1}{p^{1/2}}$

30. Since $F(t)$ has a period 2π

$$L\{F(t)\} = \frac{\int_0^T e^{-pt}F(t)\,dt}{1-e^{-pt}} = \frac{\int_0^\pi e^{-pt}\sin t\,dt + \int_\pi^{2\pi}0e^{-pt}dt}{1-e^{-2\pi p}}$$

$$=\frac{1}{1-e^{-2\pi p}}\int_0^\pi e^{-pt}\sin t\,dt = \frac{1+e^{-p\pi}}{(p^2+1)\,[1-(e^{-p\pi})^2]}$$

$$=\frac{1}{(p^2+1)\,(1-e^{-p\pi})}$$

32. $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

$$=\int_0^1 e^{-pt}0dt + \int_1^2 e^{-pt}t\,dt + \int_2^\infty e^{-pt}0dt$$

$$L\{F(t)\} = \left[-t\frac{e^{-pt}}{p} - \frac{e^{-pt}}{p^2}\right]_1^2$$

$$=-\left(\frac{2}{p}+\frac{1}{p^2}\right)e^{-2p} + \left(\frac{1}{p}+\frac{1}{p^2}\right)e^{-p}$$

So, $\lim\limits_{p\to 1} L\{F(t)\} = -\dfrac{3}{e^2} + \dfrac{2}{e}$

34. $L\{\sin x\} = \dfrac{1}{p^2+1}, p > 0$ say $f(p)$

and $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ exists and is equal to 1.

So, $$L\left\{\frac{\sin x}{x}\right\} = \int_p^\infty f(x)\,dx = \int_p^\infty \frac{1}{x^2+1}dx$$

$$= [\tan^{-1} x]_p^\infty = \cot^{-1} p$$

Since $$L\left\{\frac{\sin x}{x}\right\} = \int_0^\infty e^{-px}\frac{\sin x}{x}dx$$

$$\therefore \int_0^\infty \frac{e^{-px}\sin x}{x}dx = \cot^{-1} p$$

Putting $p = 1$, we get

$$\int_0^\infty \frac{e^{-x}\sin x}{x}dx = \cot^{-1} 1 = \frac{\pi}{4}$$

35. $\because L\ \{F'(t)\} = pL\{F(t)\} - F(0)$

$\therefore\ L\{J_1(t)\} = L\{-J_0'(t)\} = -L\{J_0'(t)\}$

$$= -\{pL\{J_0(t)\} - J_0(0)\} = -\left\{p\frac{1}{\sqrt{p^2+1}-1}\right\}$$

$$= 1 - \frac{p}{\sqrt{p^2+1}}$$

37. Let $\phi(t) = \sin t$

so $$F(t)\begin{cases}\phi\left(t-\dfrac{\pi}{3}\right), & t > \dfrac{\pi}{3}\\ 0, & t < \dfrac{\pi}{3}\end{cases}$$

So, $$L\{\phi(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = f(p)$$

By second shifting theorem we have

$$L\{F(t)\} = e^{-\left(\frac{\pi}{3}\right)p.f(p)} = \frac{e^{-\pi p/3}}{p^2+1}, p > 0$$

39. $L\{\sin at\} = \dfrac{a}{p^2+a^2}$

$$\therefore\ L\{t\sin at\} = -\frac{d}{dp}\left\{\frac{a}{p^2+a^2}\right\}$$

$$= \frac{2ap}{(p^2+a^2)^2}$$

40. Since we know that $L\left\{\dfrac{\sin t}{t}\right\} = \cos^{-1} p$

Since if $L\{F(t)\} = f(p)$ then $L\{\int_0^t F(x)\,dx\} = \dfrac{F(p)}{p}$

$$\therefore\quad L\left\{\int_0^t \frac{\sin x}{x}dx\right\} = \frac{\cot^{-1} p}{p}$$

41. Let $F(t) = e^{-t} - e^{-3t}$

$$L\{F(t)\} = \frac{1}{p+1} - \frac{1}{p+3} = F(p)$$

So, $$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty F(x)\,dx = \int_p^\infty\left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx$$

$$= \left[\log\left(\frac{x+1}{x+3}\right)\right]_p^\infty = \lim_{x\to\infty} \log\left(\frac{x+1}{x+3}\right) - \log\left(\frac{p+1}{p+3}\right)$$

$$= \lim_{x\to\infty} \log\left(\frac{1+\frac{1}{x}}{1+\frac{3}{x}}\right) - \log\left(\frac{p+1}{p+3}\right)$$

$$= -\log\left(\frac{p+1}{p+3}\right) = \log\left(\frac{p+3}{p+1}\right)$$

So, $$\int_0^\infty e^{-pt}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = \log\left(\frac{p+3}{p+1}\right)$$

$p \to 0$ we get $\displaystyle\int_0^\infty \left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = \log 3$

42. $L\ \{t\cos 4t\} = -\dfrac{d}{dp}L\ \{\cos 4t\} = -\dfrac{d}{dp}\left\{\dfrac{p}{p^2+16}\right\}$

or $$\int_0^\infty e^{-pt}t\cos 4t\,dt = \frac{p^2-16}{(p^2+16)^2}$$

put $p = 3$ we get $\displaystyle\int_0^\infty e^{-3t}t\cos 4t\,dt = \frac{7}{625}$

43. Given $L\ \{erf\sqrt{t}\} = \dfrac{1}{p\sqrt{p+1}} = F(p)$

So, $L\{e^{3t}erf\sqrt{t}\} = F(p-3)$

$$= \frac{1}{(p-3)\sqrt{p-3+1}} = \frac{1}{(p-3)\sqrt{p-2}}$$

45. Since $L\ \{t^{n-1}\} = \dfrac{(n-1)!}{p^n} = F(p)$

So, $$L\left\{\frac{e^{-at}t^{n-1}}{(n-1)!}\right\} = \frac{1}{(n-1)!}L\{e^{-at}\ t^{n-1}\}$$

$$= \frac{1}{(n-1)!}f(p+a) = \frac{1}{(n-1)!}.\frac{(n-1)!}{(p+a)^n}$$

$$= \frac{1}{(p+a)^n}$$

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Unit-IV

CHAPTER

15

The Inverse Laplace Transform

INVERSE LAPLACE TRANSFORM

1. If $L\{F(t)\} = f(p)$ then $F(t)$ is called an inverse Laplace transform of $f(p)$ and is written as $F(t) = L^{-1}\{f(p)\}$

   Here L^{-1} is called the inverse Laplace transformation operator.

2. If $f_1(p)$ and $f_2(p)$ be the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively and C_1, C_2 are two constants, then

   $$L^{-1}\{C_1 f_1(p) + C_2 f_2(p)\} = C_1 L^{-1}\{f_1(p)\} + C_2 L^{-1}\{f_2(p)\}$$

   This is called Linearity property.

3. The inverse Laplace transforms of some functions are

   (i) $L^{-1}\left\{\frac{1}{p}\right\} = 1$

   (ii) $L^{-1}\left\{\frac{1}{p^2}\right\} = t$

   (iii) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{t(n+1)}$ or $\frac{t^n}{n!}$

   (iv) $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

   (v) $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin at}{a}$

   (vi) $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

   (vii) $L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{\sin hat}{a}$

   (viii) $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cos hat$

PROPERTIES OF INVERSE LAPLACE TRANSFORMS

1. If $L^{-1}\{f(p)\} = F(t)$ then

   $$L^{-1}\{f(p-a)\} = e^{at}F(t) = e^{at}L^{-1}\{f(p)\}$$

   This is called first translation or shifting theorem.

2. If $L^{-1}\{f(p)\} = F(t)$

   $$\text{where } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

   This is called second translation or shifting theorem.

3. If $L^{-1}\{f(p)\} = F(t)$ then

   $$L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

   This is called change of scale property.

   If $L^{-1}\{f(p)\} = F(t)$ then

   $$L^{-1}\{f^n(p)\} = L^{-1}\left\{\frac{d^n}{dp^n}f(p)\right\} = (-1)^n t^n F(t)$$

4. If $L^{-1}\{f(p)\} = F(t)$ then

   $$L^{-1}\{\int_p^\infty f(x)\,dx\} = \frac{F(t)}{t}$$

5. If $L^{-1}\{f(p)\} = F(t)$ and $F(0) = 0$ then

   $$L^{-1}\{pf(p)\} = F'(t)$$

6. If $F(t)$ is sectionally continuous and of exponential order a and such that $\lim_{t\to 0}\frac{F(t)}{t}$ exists then for $p > a$

   $$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(x)\,dx$$

   Continuing it we have

   $$L^{-1}\left[\frac{f(p)}{p^n}\right] = \int_0^t\int_0^t\int_0^t - \int_0^t F(t)\,dt^n$$

7. If $F(t)$ and $G(t)$ be two functions of class A, then the convolution of two functions $F(t)$ and $G(t)$ denoted by $F * G$ is defined as

$$F * G = \int_0^t F(x)\,G(t-x)\,dx$$

$F \times G$ is commutative, Associative and distributive w.r.t. addition.

8. If $F(t)$ and $G(t)$ be two functions of class A and let $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$ then

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(x)\,G(t-x)\,dx = F * G$$

This is called convolution theorem.

9. If $F(p)$ and $G(p)$ be two polynomials in p where $F(p)$ has degree less than that of $G(p)$ and $G(p)$ has n distinct factors (zeros)

$$\alpha_r,\ r = 1, 2, 3, \ldots n$$

i.e. $$G(p) = (p-\alpha_1)(p-\alpha_2)\ldots(p-\alpha_n)$$

then $$L^{-1}\left[\frac{F(p)}{G(p)}\right] = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)}\, e^{\alpha_r t}$$

This is called Heaviside's Expansion theorem.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$ is equal to :

(a) $\frac{1}{\sqrt{t}}$ (b) $\sqrt{\frac{1}{t}}$ (c) $\sqrt{\pi t}$ (d) $\frac{1}{\sqrt{\pi t}}$

2. $L^{-1}\left\{\frac{1}{p^2+4}\right\}$ is :

(a) $\frac{\cos 2t}{2}$ (b) $\frac{\cos h2t}{2}$ (c) $\frac{\sin 2t}{2}$ (d) $\frac{\sin h2t}{2}$

3. The value of $L^{-1}\left\{\frac{4}{(p-2)}\right\}$ is :

(a) e^{2t} (b) $4e^{2t}$ (c) $2e^{2t}$ (d) $\frac{e^{2t}}{4}$

4. If $L^{-1}\left\{\frac{e^{-\frac{1}{p}}}{\sqrt{p}}\right\} = \frac{\cos(2\sqrt{t})}{\sqrt{\pi t}}$ then $L^{-1}\left\{\frac{e^{-a/p}}{\sqrt{p}}\right\}$ for $a > 0$ is :

(a) $\frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi t}}$ (b) $\frac{\cos 2\sqrt{\frac{k}{t}}}{\sqrt{\pi t}}$

(c) $\frac{\cos 2\sqrt{kt}}{\sqrt{\pi t}}$ (d) None of these

5. $L^{-1}\left\{\frac{e^{-p}}{p^2}\right\}$ is equal to :

(a) $(t-1)\,e^{-t}$ (b) $(t+1)\,e^{-t}$

(c) $(t-1)\,H(t-1)$ (d) $(t+1)+(1)$

6. $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$ is equal to :

(a) $e^t \cos t$ (b) $e^{3t} \sin t$

(c) $e^{3t} \cos t$ (d) $e^t \sin t$

7. The value of $L^{-1}\{(2p+3)^{-1/2}\}$ is :

(a) $\frac{1}{e^{\frac{3}{2}t}.\sqrt{2\pi t}}$ (b) $\frac{e^{3/2t}}{\sqrt{\pi t}}$

(c) $\frac{e^{\frac{3}{2}t}}{\sqrt{2\pi t}}$ (d) $\frac{1}{e^{\frac{3}{2}t}\sqrt{\pi t}}$

8. If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t \sin t}{2}$ then $L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$ is :

(a) $\frac{t}{2}\sin\frac{t}{2}$ (b) $\frac{t}{3}\sin\frac{t}{3}$

(c) $\frac{t}{4}\sin\frac{t}{4}$ (d) $\frac{\sin\frac{t}{2}}{\frac{t}{2}}$

9. The inverse Laplace transform of $\frac{e^{-3p}}{p^3}$ is :

(a) $(t-3)\,H(t-3)$ (b) $tH(t-3)$

(c) $(t-3)^2 H(t-3)$ (d) $\frac{(t-3)^2}{2} H(t-3)$

10. The value of $L^{-1}\left\{\frac{pe^{-ap}}{p^2-w^2}\right\}$ is :

(a) $(t-a)\,H(t-a)$ (b) $\cos w\,(t-a)\,H(t-a)$

(c) $\cos hw(t-a)\,H(t-a)$ (d) None of these

11. $L^{-1}\left\{\frac{1}{p(p+3)}\right\}$ is equal to :

(a) $1+e^{-3t}$ (b) $1-e^{-3t}$

(c) $\frac{e^{-3t}-1}{3}$ (d) $\frac{1-e^{-3t}}{3}$

12. $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$ is :

(a) $\frac{2(1-\cos ht)}{t}$ (b) $\frac{1-\sin ht}{t}$

(c) $\frac{1+\cos ht}{t}$ (d) None of these

13. $L^{-1}\left\{\frac{1}{p^2(p+1)}\right\}$ is :

(a) $t+e^{-t}$ (b) $t-1+e^{-t}$

(c) $t+1+e^{-t}$ (d) $t-e^{-t}$

14. $L^{-1}\left\{\log\frac{p+2}{p+1}\right\}$ is :

(a) $\frac{e^t+e^{2t}}{t}$ (b) $\frac{e^t+e^{-2t}}{t}$

(c) $\frac{e^{-t}-e^{-2t}}{t}$ (d) $\frac{e^{-t}+e^{-2t}}{t}$

15. $L^{-1}\left\{\frac{1}{(p-1)(p+3)}\right\}$ is equal to :

(a) $\frac{e^t-e^{-2t}}{3}$ (b) $\frac{e^t+e^{-2t}}{3}$

(c) $\frac{e^{-t}-e^{-2t}}{2}$ (d) None of these

16. The value of $L^{-1}\left\{\frac{19p+37}{(p+1)(p-2)}\right\}$ is :

(a) $3e^{2t}-5e^{-t}$ (b) $3e^{2t}+5e^{-t}$

(c) $5e^{2t}+3e^{-t}$ (d) $5e^{2t}-3e^{-t}$

17. $\int_0^\infty e^{-x^2}dx$ is equal to :

(a) $\frac{\pi}{2}$ (b) π

(c) $\sqrt{\pi}$ (d) $\frac{\sqrt{\pi}}{2}$

18. The value of $L^{-1}\left\{\frac{2p+1}{p(p+1)}\right\}$ is :

(a) $\sin t+1$ (b) $\cos t+1$ (c) $e^{-t}+1$ (d) $\frac{\sin^2 t}{2}$

19. $L^{-1}\left\{-\frac{3}{9p^2-16}\right\}$ is equal to :

(a) $\frac{1}{4}\cos h\frac{4t}{3}$ (b) $-\frac{1}{4}\sin h\frac{4t}{3}$

(c) $\frac{1}{4}\sin h\frac{4t}{3}$ (d) $-\frac{1}{4}\cos h\frac{4t}{3}$

20. $L^{-1}\left\{\frac{1}{(p+a)^n}\right\}$ is :

(a) $\frac{e^{-at}t^{n-1}}{(n-1)!}$ (b) $\frac{e^{-at}t^n}{n!}$

(c) $\frac{e^{-at}t^{n+1}}{(n-1)!}$ (d) None of these

21. The inverse Laplace transform of $\frac{e^{-p\pi}}{p^2+1}$ is :

(a) $(t-\pi)H(t-\pi)$ (b) $tH(t-\pi)$

(c) $-\sin t H(t-\pi)$ (d) $\cos t H(t-\pi)$

22. $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$ is :

(a) $\frac{2(1-\cos t)}{t}$ (b) $\frac{1-\sin t}{t}$

(c) $\frac{\log(1+t)}{t}$ (d) None of these

23. If $L^{-1}\{f(p)\}=F(t)$ then $L^{-1}\{f(p-a)\}$ is :

(a) $e^{-at}F(t)$ (b) $e^{at}F(t)$

(c) $\frac{1}{a}F\left(\frac{t}{a}\right)$ (d) $(-t)^nF(t)$

24. If $L^{-1}\{f(p)\}=F(t)$ then $L^{-1}\left\{\frac{f(p)}{p}\right\}$ is :

(a) $\int_0^x\frac{F(x)}{x}dx$ (b) $\int_0^x F(x)\,dx$

(c) $F'(t)$ (d) $(-1)^n t^n F(t)$

25. The value of $\int_0^t \sin x\cos(t-x)\,dx$ is equal to :

(a) $\sin t+\cos t$ (b) $t\sin t-1$

(c) $t\cos t+1$ (d) $\frac{t\sin t}{2}$

26. By convolution, the value of $|*||*|x|$ (n times) is :

(a) $\frac{t}{(n-1)!}$ (b) t^n (c) $\frac{t^n}{n!}$ (d) $\frac{t^{n-1}}{(n-1)!}$

27. The value of $L^{-1}\left\{\frac{p}{p^2+36}\right\}$ is :

(a) $\sin 6t$ (b) $\sin h\ 6t$

(c) $\cos 6t$ (d) $\cos h\ 6t$

28. $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\}$ is :

(a) $1-\frac{t^2}{(L^2)^{2+\dots}}$ (b) $1-\frac{t^2}{L^2}+\dots$

(c) $\frac{1+t^2}{(L^2)^2}+\dots$ (d) $\frac{1+t^2}{L^2}+\dots.$

29. $L^{-1}\left[\frac{1}{(p-a)^3}\right]$ is equal to :

(a) $\frac{e^{at}}{2}$ (b) $\frac{te^{at}}{2}$ (c) $\frac{t^2e^{at}}{2}$ (d) $\frac{t^3e^{at}}{2}$

30. The value of $L^{-1}\left\{\log\frac{p+3}{p+2}\right\}$ is :

(a) $\frac{e^{-2t}+e^{-3t}}{2}$ (b) $\frac{e^{-2t}-e^{-3t}}{t}$

(c) $\frac{e^{-2t}+e^{-3t}}{t}$ (d) $\frac{e^{-2t}-e^{-3t}}{2}$

31. If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}=\frac{t\sin t}{2}$ then $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$ is :

(a) $\sin t + t\cos t$ (b) $\frac{\sin t + t\cos t}{2}$

(c) $\frac{t\sin t - \cos t}{2}$ (d) $\frac{\sin t - t\cos t}{2}$

32. The value of $L^{-1}\left\{\frac{1}{(p+1)\,(p-2)}\right\}$ is :

(a) $\frac{e^{2t}-e^{-t}}{3}$ (b) $\frac{e^{t}-e^{-t}}{3}$

(c) $\frac{e^{2t}+e^{-t}}{3}$ (d) $\frac{e^{t}+e^{-t}}{3}$

33. $L^{-1}\left\{\frac{p+5}{p^2+1}\right\}$ is equal to :

(a) $\sin t - 2\cos t$ (b) $3\sin t - 2\cos t$

(c) $3\sin t + \cos t$ (d) $2\sin t - 3\cos t$

34. If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{f(ap)\}$ is :

(a) $\frac{1}{a}F\left(\frac{t}{a}\right)$ (b) $aF\left(\frac{t}{a}\right)$

(c) $atF(t)$ (d) $\frac{1}{a}F(at)$

35. $L^{-1}\left\{\frac{e^{-\pi p}}{p^2+1}\right\}$ is equal to :

(a) $\sin t\, H\,(t-\pi)$ (b) $\cos t\, H(t-\pi)$

(c) $t\sin(t-\pi)$ (d) $\sin(t-\pi)\ H\ (t-\pi)$

36. If $L^{-1}\{f(p)\} = F(t)$ and $F(0) = 0$ then $L^{-1}\{pf(p)\}$ is :

(a) $\frac{F(t)}{t}$ (b) $tF(t)$

(c) $\int_0^x F(x, dx)$ (d) $F'(t)$

37. If $L^{-1}\left\{\frac{e^{-\frac{1}{2}}}{\sqrt{p}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ then $L^{-1}\left\{\frac{e^{-ap}}{\sqrt{p}}\right\}$ is :

(a) $\frac{\cos\,(2\sqrt{at})}{\sqrt{\pi t}}$ (b) $\frac{\cos\,(2\sqrt{\pi t})}{\sqrt{at}}$

(c) $\frac{1}{a}\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ (d) None of these

38. $L^{-1}\left\{\frac{e^{-ap}}{p^2}\right\}$, $a > 0$ is :

(a) at when $t > a$ and 0 otherwise

(b) $(t-a)$ when $t > a$ and 0 otherwise

(c) $(t+a)$ when $t > a$ and 0 otherwise

(d) None of these

39. The value of $L^{-1}\left\{\frac{3p+7}{p^2+2p-3}\right\}$ is :

(a) $4e^{3t}-e^{-t}$ (b) $e^{3t}+e^{-2t}$

(c) $4e^{-3t}-e^{t}$ (d) $2e^{3t}-e^{-t}$

40. The value of $L^{-1}\left\{\frac{1}{p(p+1)}\right\}$ is :

(a) e^{-t} (b) e^{t} (c) $1+e^{-t}$ (d) $1-e^{-t}$

41. If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\left\{\frac{1}{2}F\left(\frac{p}{2}\right)\right\}$ is :

(a) $F\left(\frac{t}{2}\right)$ (b) $\frac{1}{2}F\left(\frac{t}{2}\right)$

(c) $F\ (2t)$ (d) $\frac{1}{2}F(t)$

42. $L^{-1}\{pf(p)-F(0)\}$ is equal to :

(a) $F(t)$ (b) $F'(t)$ (c) $F''(t)$ (d) $tF'(t)$

43. $L^{-1}\left\{\frac{1}{9p^2+6p+1}\right\}$ is equal to :

(a) $\frac{te^{-t/3}}{9}$ (b) $\frac{te^{t/3}}{3}$

(c) $\frac{te^{-t/2}}{9}$ (d) None of these

44. The value of $L^{-1}\left\{\frac{1}{2(p-1)^2+32}\right\}$ is :

(a) $\frac{e^{4t}\sin t}{2}$ (b) $\frac{e^{t}\sin 8t}{4}$

(c) $\frac{e^{t}\cos 4t}{8}$ (d) $\frac{e^{t}\sin 4t}{8}$

45. $L^{-1}\left\{\frac{2p-6}{p^2-3p+2}\right\}$ is equal to :

(a) $4e^{t}-2e^{2t}$ (b) $4e^{t}-\frac{e^{2t}}{2}$

(c) $\frac{4e^{t}+e^{2t}}{2}$ (d) $\frac{3e^{2t}}{2}$

46. $L^{-1}\left\{\tan^{-1}\frac{1}{p}\right\}$ is :

(a) $\frac{\tan t}{t}$ (b) $\frac{\cos t}{t}$ (c) $\frac{\sec t}{t}$ (d) $\frac{\sin t}{t}$

47. The value of $L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\}$ is :

(a) $\sin t$ (b) $t+\sin t$ (c) $t-\sin t$ (d) $t-\sin t$

48. $L^{-1}\left\{\frac{1}{p(p+1)^2}\right\}$ is :

(a) $e^{-t}(t+1)$ (b) $1-e^{-t}(t+1)$

(c) $1+e^{-t}$ (d) $e^{t}(t-1)$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(c)	**3.**	(b)	**4.**	(a)	**5.**	(c)	**6.**	(b)	**7.**	(a)	**8.**	(c)	**9.**	(d)	**10.**	(c)
11.	(d)	**12.**	(a)	**13.**	(b)	**14.**	(c)	**15.**	(a)	**16.**	(d)	**17.**	(d)	**18.**	(c)	**19.**	(b)	**20.**	(a)
21.	(c)	**22.**	(a)	**23.**	(b)	**24.**	(b)	**25.**	(d)	**26.**	(d)	**27.**	(c)	**28.**	(a)	**29.**	(c)	**30.**	(b)
31.	(d)	**32.**	(a)	**33.**	(b)	**34.**	(a)	**35.**	(d)	**36.**	(d)	**37.**	(a)	**38.**	(b)	**39.**	(a)	**40.**	(d)
41.	(c)	**42.**	(b)	**43.**	(a)	**44.**	(d)	**45.**	(a)	**46.**	(d)	**47.**	(d)	**48.**	(b)				

HINTS AND SOLUTIONS

1. $$L^{-1}\left(\frac{1}{\sqrt{p}}\right)=L^{-1}\left\{\frac{1}{p^{1/2}}\right\}=\frac{t^{\frac{1}{2}-1}}{t\left(\frac{1}{2}\right)}=\frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}=\frac{1}{\sqrt{t\pi}}$$

4. Given that $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$

$$\therefore \qquad L^{-1}\left\{\frac{e^{-\frac{1}{pk}}}{(pk)^{1/2}}\right\}=\frac{1}{k}\frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\frac{\pi t}{k}}}$$

or $$L^{-1}\left\{\frac{e^{-\frac{1}{kp}}}{\sqrt{p}}\right\}=\frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi t}}$$

put $k=\frac{1}{a}$ we get

$$L^{-1}\left\{\frac{e^{-\frac{a}{p}}}{\sqrt{p}}\right\}=\frac{\cos 2\sqrt{at}}{\sqrt{\pi f}}$$

6. $$L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}=L^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$$

$$e^{3t}L^{-1}\left\{\frac{1}{p^2+1}\right\}=e^{3t}\sin t$$

8. Given that $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}=\frac{t\sin t}{2}$

We have $L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}=\frac{1}{2}\frac{1}{a}\frac{t}{a}\sin\frac{t}{a}$

or $$L^{-1}\left\{\frac{2a^2p}{(a^2p^2+1)^2}\right\}=\frac{t}{a}\sin\frac{t}{a}$$

put $a=4$, we get

$$L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}=\frac{t}{4}\sin\frac{t}{4}$$

10. Since $L^{-1}\left\{\frac{p}{p^2-\omega^2}\right\}=\cos h\omega t$

$$L^{-1}\left\{\frac{pe^{-ap}}{p^2-\omega^2}\right\}=\cos h\omega(t-a)\,H\,(t-a)$$

12. Let $f(p)=\log\left(1-\frac{1}{p^2}\right)$

$$=\log\left(\frac{p^2-1}{p^2}\right)=-2\log p+\log(p^2-1)$$

$$\therefore \qquad f(p)=-2\left(\frac{1}{p}-\frac{p}{p^2-1}\right)$$

$$\therefore \qquad L^{-1}\{f(p)\}=-2\,(1-\cos ht)$$

or $$-tL^{-1}\{f(p)\}=-2\,(1-\cos ht)$$

or $$L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}=\frac{2}{t}(1-\cos ht)$$

14. Let $f(p)=\log\left(\frac{p+2}{p+1}\right)=\log(p+2)-\log(p+1)$

$$f'(p)=\frac{1}{p+2}-\frac{1}{p+1}$$

$$\therefore L^{-1}\{f'(p)\} = e^{-2t} - e^{-t}$$

or $$= tL^{-1}\{f(p)\} = e^{-2t} - e^{-t}$$

$$\therefore\ L^{-1}\{f(p)\} = L^{-1}\left\{\log\frac{p+2}{p+1}\right\} = \frac{1}{t}(e^{-t} - e^{-2t})$$

16. Given $L^{-1}\left\{\dfrac{19p+37}{(p+1)\,(p-2)}\right\}$

Here $F\,(p) = 19p + 37$

$$G\,(p) = (p+1)\,(p-2) = p^2 - p - 2$$

Here $G\,(p)$ has 3 distinct zeroes,

$$\alpha_1 = -1, \alpha_2 = 2$$

$$G'(p) = 2p - 1$$

By Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{19p+37}{(p+1)\,(p-2)}\right\} = \frac{F(-1)}{G'(-1)}e^{-t} + \frac{F(2)}{G'(2)}e^{2t} = -3e^{-t} + 5e^{2t}$$

17. Let $F(t) = \int_0^\infty e^{-tx^2 dx}$

$$L\{F(t)\} = \int_0^\infty L\{e^{-tx^2}\}\ dx$$

$$= \int_0^\infty \frac{dx}{p+x^2} = \left(\frac{1}{\sqrt{p}}\tan^{-1}\frac{x}{\sqrt{p}}\right)_0^\infty$$

$$= \frac{\pi}{2\sqrt{p}}$$

$$\therefore \qquad F(t) = L^{-1}\left\{\frac{\pi}{2\sqrt{p}}\right\} = \frac{\pi}{2}\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

or $\int_0^\infty e^{-tx^2}.dx = \dfrac{1}{2}\sqrt{\dfrac{\pi}{t}}$

put $t = 1$ we get

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

21. Since $L^{-1}\left\{\dfrac{1}{p^2+1}\right\} = \sin t$

$$L^{-1}\left\{\frac{e^{-p\pi}}{p^2+1}\right\} = \sin\,(t-\pi)\ H\ (t-\pi)$$

$$= -\sin t\ H(t-\pi)$$

25. Let. $F(t) = \int_0^t \sin x\cos\,(t-x)\,dx$

By convolution theorem we have

$$L\ \{F(t)\} = L\{\sin t\}\,L\{\cos t\}$$

$$= \frac{1}{p^2+1}.\frac{p}{p^2+1} = \frac{p}{(p^2+1)^2}$$

$$\therefore \qquad F(t) = L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t\sin t}{2}$$

26. Since $F * G = \int_0^t F(x)G\,(t-x)\,dx$

$$|*| = \int_0^t 1.1dx = t$$

$$|*|*| = t * 1 = \int_0^t x.1\,dx = \frac{t^2}{2}$$

$$|*|*|*| = \left(\frac{t^2}{2}\right) * 1 = \int_0^1 \frac{x^2}{2}1dx = \frac{t^3}{3!}$$

Proceeding similarly, we have

$$|*|*|\,....*|\ (n \text{ times}) = \frac{t^{n-1}}{(n-1)!},\ n = 1, 2, 3,...$$

31. Given $L^{-1}\left\{\dfrac{p}{(p^2+1)^2}\right\} = \dfrac{t\sin t}{2} = F(t)$

$$\therefore\ L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{p}.\frac{p}{(p^2+1)^2}\right\}$$

$$= \int_0^t F(x)\,dx = \frac{1}{2}\int_0^t x\sin x\,dx = \frac{\sin t - t\cos t}{2}$$

35. $\because L^{-1}\left\{\dfrac{1}{p^2+1}\right\} = \sin t$

$$L^{-1}\left\{\frac{e^{-p\pi}}{(p^2+1)}\right\} = \sin\,(t-\pi)H\,(t-\pi)$$

$$= -\sin t H\,(t-\pi)$$

37. Since $L^{-1}\left\{\dfrac{e^{-\frac{1}{p}}}{p^{1/2}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$

$$\therefore\ L^{-1}\left\{\frac{e^{-\frac{1}{pk}}}{(pk)^{1/2}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\frac{\pi t}{k}}}$$

or $$L^{-1}\left\{\frac{e^{-\frac{1}{pk}}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\pi t}}$$

Put $k = \dfrac{1}{a}$ we get $L^{-1}\left\{\dfrac{e^{-\frac{a}{p}}}{\sqrt{p}}\right\} = \dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$

38. $L^{-1}\left\{\dfrac{1}{p^2}\right\} = t$

So, $$L^{-1}\left\{\frac{e^{-ap}}{p^2}\right\} = (t-a)H\,(t-a) = \begin{Bmatrix} t-a & t > a \\ 0 & t \le a \end{Bmatrix}$$

43. $L^{-1}\left\{\dfrac{1}{(9p^2+6pt)}\right\}$

$$= L^{-1}\left\{\frac{1}{(3p+1)^2}\right\} = \frac{1}{9}L^{-1}\left\{\frac{1}{\left(p+\frac{1}{3}\right)^2}\right\}$$

$$= \frac{1}{9}e^{-t/3}L^{-1} = \left\{\frac{1}{p^2}\right\} = -\frac{t}{9}\,e^{-t/3}$$

44. $L^{-1}\left\{\dfrac{1}{2(p-1)^2+3^2}\right\} = e^t L^{-1}\left\{\dfrac{1}{2p^2+3^2}\right\}$

$$= \frac{e^t}{2}L^{-1}\left\{\frac{1}{p^2+16}\right\} = \frac{e^t}{2}\,\frac{\sin 4t}{4} = \frac{e^t\sin 4t}{8}$$

45. Here $F(p) = 2p-6$

and $G(p) = p^2-3p+2$

$$G(p) = (p-1)\,(p-2)$$

$\therefore\quad G'(p) = 2p-3$

By Heaviside expansion formula we have

$$L^{-1}\left\{\frac{2p-6}{p^2-3p+2}\right\} = \frac{F(1)}{G'(1)}e^t + \frac{F(2)}{G'(2)}e^{2t}$$

$$= \frac{-4}{-1}e^t + \frac{-2}{1}e^{2t}$$

$$= 4e^t - 2e^{2t}$$

46. Let $f(p) = \tan^{-1}\left(\dfrac{1}{p}\right) = \cot^{-1}p$

$$f'(p) = -\frac{1}{p^2+1}$$

So $L^{-1}\{f'(p)\} = tL^{-1}\left\{-\dfrac{1}{p^2+1}\right\}$

$$f'(p) = -\sin t$$

But $L^{-1}\{f'(p)\} = -tL^{-1}\{f(p)\}$

$\therefore\quad -tL^{-1}\{f(p)\} = -\sin t$

$$L^{-1}\{f(p)\} = \frac{\sin t}{t}$$

or $L^{-1}\left\{\tan^{-1}\left(\dfrac{1}{p}\right)\right\} = \dfrac{\sin t}{t}$

47. $L^{-1}\left\{\dfrac{1}{(p^2+1)}\right\} = \sin t$

$$L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = \int_0^t \sin x\,dx = 1-\cos t$$

$$L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\} = \int_0^t (1-\cos x)\,dx = t-\sin t$$

48. $L^{-1}\left\{\dfrac{1}{p(p+1)^2}\right\} = L^{-1}\left\{\dfrac{1}{(p+1-1)\,(p+1)^2}\right\}$

$$= e^{-t}L^{-1}\left\{\frac{1}{(p-1)\,p^2}\right\}$$

Now, $L^{-1}\left\{\dfrac{1}{p-1}\right\} = e^t$

$\therefore\quad L^{-1}\left\{\dfrac{1}{p(p-1)}\right\} = \displaystyle\int_0^t e^x\,dx = e^t-1$

$$L^{-1}\left\{\frac{1}{p^2(p-1)}\right\} = \int_0^t (e^x-1)\,dx = e^t-t-1$$

$\therefore\ L^{-1}\left\{\dfrac{1}{p(p+1)^2}\right\} = e^{-t}(e^t-t-1) = 1-e^{-t}(t+1)$

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Unit-IV

CHAPTER

16

Applications of Laplace Transform

SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

1. The Laplace transform is very useful in solving ordinary linear differential equations with constant coefficient.

   Consider a linear differential equation with constant coefficients.

$$\frac{d^n y}{dt^n} + A_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + A_{n-1} \frac{dy}{dt} + A_n y = F(t) \qquad \ldots(1)$$

   Where $F(t)$ is a function of independent variable t.

   Let $y(0) = C_0,\ y(0) = C_1, \ldots, y^{n-1}(0) = C_{n-1}$ ...(2)

   be the given initial or boundary conditions.

   Where $C_0, C_1, C_2, \ldots, C_{n-1}$ are all constants.

   To solve equation (1) we take the Laplace transform of both sides of equation (1) and using contition (2) we obtain an algebraic equation called subsidiary equation form which

   $\bar{y}(p) = L\{y(t)\}$ is determined. The required solution can be obtained by finding the inverse Laplace transform of $\bar{y}(p)$.

2. The Laplace transform is very useful in solving the differential equations having the terms $t^m y^n(t)$ is variable coefficients, where Laplace transform is

$$L\{t^m y^n(t)\} = (-1)^m \frac{dm}{dp^m} [L\{y^n(t)\}]$$

3. The Laplace transform can also be used in solving two or more simultaneous ordinary differential equations.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

1. Laplace transform is also useful in solving partial differential equations when the boundary conditions are given

   If $y(x, t)$ is a function of x and t then

   (i) $L\left\{\frac{\partial y}{\partial t}\right\} = p\bar{y}(x, p) - y(x, 0)$

   (ii) $L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = p^2\bar{y}(x, p) - py\,(x, 0) - yt(x, 0)$

   (iii) $L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d\bar{y}}{dx}$

   (iv) $L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2\bar{y}}{dx^2}$

   where, $L\{y(x, t)\} = \bar{y}(x, p)$

SOLUTION OF INTEGRAL EQUATIONS

1. Laplace transform is also useful in solving various integral equations such as

$$F(t) = y(f) + \int_a^b k(u, t) F(u)\, dx$$

   where $y(t)$ and $k(u, t)$ are known a and b are either constants or functions of t. Here the function $F(t)$ which appears under the integral sign is to be determined.

2. An equation of the form $G(t) = \int_0^t \frac{F(u)\, du}{(t-u)^n}$

   is called Abel's integral equation with $r < n < 1$.

3. An integral equation of the form

$$F(t) = y(t) + \int_0^t k(t-u)\, F(x)\, dx \text{ or}$$

   $F(t) = y(t) + k(t) * F(t)$ is called an integral equation of convolution type.

4. An integral equation in which various derivatives of the unknown function $F(t)$ can also be present is called integer-differential equation such as

$$F'(t) = F(t) + y(t) + \int_0^t k(t-u)\, F(u)\, du$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. If $y(x, t)$ is a functio.n of x and t then $L\left\{\frac{\partial y}{\partial t}\right\}$ is :

(a) $\bar{y}(x,p) - y(x, 0)$ (b) $\frac{d\bar{y}}{dt}$

(c) $p\bar{y}(x, p) - y(x, 0)$ (d) $p^2\bar{y}(x, p) - py(x, 0)$

2. The solution of $(D + 1)\, y = 1$, $y = 2$ when $t = 0$ is :

(a) $1 + e^t$ (b) $1 + e^{-t}$ (c) $1 - e^{-t}$ (d) $1 - e^t$

3. The solution of $(D^2 + 1)\, y = 0$, $y(0) = 0$, $y'(0) = 1$ is :

(a) $\cos t$ (b) $e^t + 1$ (c) $\sin t$ (d) e^{-t}

4. If we take Laplace transform of $(D + 2)^2 y = 4e^{-2t}$ with $y(0) = -1$, $y'(0) = 4$, $L\{y\}$ is equal to :

(a) $\frac{4 - 2p - p^2}{(p + 2)^3}$ (b) $\frac{2p^2 + 1}{(p + 2)^3}$

(c) $\frac{2p^2 - 2p + 1}{(p + 2)^3}$ (d) None of these

5. The solution of $D^3 y = 0$ with $y(0) = y'(0) = 0$, $y''(0) = 1$ is :

(a) t (b) t^2 (c) $\frac{t^2}{2}$ (d) $\frac{t^2 + 1}{2}$

6. The solution of $(D^2 + D)\, x = 2$, $x(0) = 3$, $x'(0) = 1$ is :

(a) $1 + 2t + e^t$ (b) $2 + t - e^{-t}$

(c) $1 + 2t + e^{-t}$ (d) $2 + 2t + e^{-t}$

7. The Laplace transform of $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$, $y(x, 0) = 6e^{-3x}$ is :

(a) $\frac{d\bar{y}}{dx} - (2p + 1)\,\bar{y} = -12e^{-3x}$

(b) $\frac{d\bar{y}}{dx} - p\bar{y} = 12e^{-3x}$

(c) $\frac{d\bar{y}}{dx} + 2p\bar{y} = e^{-3x}$ (d) None of these

8. The solution of $(D^2 + 1)\, y = 0$ under the condition $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$, is:

(a) $e^t + 1$ (b) $\cos t + 1$ (c) $\sin t - 1$ (d) $\cos t$

9. For $(D^2 - D - 6)\, y = 2$, $t > 0$ with $y(0) = 1$, $y'(0) = 0$. $L\{y\}$ is equal to :

(a) $\frac{p^2 - p + 2}{p(p^2 - p - 6)}$ (b) $\frac{p^2 + 1}{(p^2 - p - 6)}$

(c) $\frac{1}{p\,(p^2 + 1)}$ (d) $\frac{p + 1}{p^2 - p - 6}$

10. The Laplace transform of $3\frac{\partial^2 y}{dx^2} = \frac{\partial y}{\partial t}$ with $y(x, 0) = 30\cos 5x$ is :

(a) $(D^2 + 3)\,\bar{y} = \cos 5x$

(b) $\left(D^2 - \frac{p}{3}\right)\bar{y} = -10\cos 5x$

(c) $\left(D^2 + \frac{p}{3}\right)\bar{y} = 10\cos 5x$

(d) $\left(D^2 - \frac{p}{3}\right)\bar{y} = \cos 5x$

11. The solution of $F(t) = a\sin t - 2\int_0^t F(u)\cos(t - 9)\,du$ is :

(a) e^{-t} (b) te^t (c) ate^{-t} (d) at^2e^t

12. The solution of $F(t) = 1 + 2\int_0^t F(t - u)\,e^{-2u}\,du$ is :

(a) $2t$ (b) $1 + 2t$ (c) $1 - 2t$ (d) $2t + t$

13. $L\left\{\frac{\partial y}{\partial x}\right\}$ is equal to :

(a) $\frac{d\bar{y}}{dx}$ (b) $p\bar{y}\,(x, p)$

(c) $p\bar{y}(x, p) - y\,(x)\,dy$ (d) $\bar{y}\,(x, p)$

14. The solution of $D^2 y = t$ when $y(0) = y'(0) = 0$ is :

(a) $\frac{t^2}{3}$ (b) $\frac{t^3}{6}$ (c) $\frac{t}{6}$ (d) Not exist

15. $L^{-1}\{p\bar{y}(x, p) - y\,(x, 0)\}$ is :

(a) $\frac{dy}{dt}$ (b) $\frac{dy}{dx}$ (c) $\frac{\partial y}{\partial x}$ (d) $\frac{\partial y}{\partial t}$

16. $L^{-1}\left\{\frac{d^2\bar{y}}{dx^2}\right\}$ is equal to :

(a) $\frac{d^2 y}{dx^2}$ (b) $\frac{d^2\bar{y}}{dx^2}$ (c) $\frac{\partial^2 y}{\partial x^2}$ (d) $\frac{\partial^2\bar{y}}{\partial x^2}$

17. The solution of $y'' + y = t$ with $y'(0) = 1$, $y\,(\pi) = 0$ is :

(a) $t\cos t + \pi$ (b) $\pi\sin t + t$

(c) $t\sin t + \pi$ (d) $\pi\cos t + t$

18. $L\left(\frac{\partial y}{\partial x}\right)$ is equal to :

(a) $\bar{y}(x,p) - y(x, 0)$ (b) $\frac{d\bar{y}}{dx}$

(c) $p\bar{y}(x,p) - y\,(x, 0)$ (d) $\bar{y}(x, p)$

19. The Laplace transform of $\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$ with $y(x, 0) = 3\sin 2\pi x$ is :

(a) $(D^2 + p)\,\overline{y} = 3\sin 2\pi x$
(b) $(D + p)\,\overline{y} = -3\sin 2\pi x$
(c) $(D - p)\,\overline{y} = 3\sin 2\pi x$
(d) $(D^2 - p)\,\overline{y} = -3\sin 2\pi x$

20. The solution of $F(t) = 4t - 3\int_0^t F(u)\sin(t-u)\,du$ is :
(a) $t + \frac{3}{2}\sin 2t$ (b) $t - \frac{3}{4}\cos 2t$
(c) $t + \sin 3t$ (d) $t - \frac{1}{2}\sin 3t$

21. The solution of $y(t) = 1 + \int_0^t y(u)\,(t-u)\,du$:
(a) $\sin t$ (b) $\cos t$ (c) $\sin ht$ (d) $\cos ht$

22. $L^{-1}\{p^2\overline{y}(x, p) - py\,(x, 0) - y + (x, 0)\}$ is :
(a) $\frac{d^2\overline{y}}{dx^2}$ (b) $\frac{\partial^2\overline{y}}{\partial x^2}$ (c) $\frac{d^2 y}{dt^2}$ (d) $\frac{\partial^2 y}{\partial t^2}$

23. The solution of $2\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$ with $y(x, 0) = 0$, $\overline{y} = L\{y\}$ is :
(a) $\overline{y} = C_1 e^{px/2} + C_2 e^{-px/2}$
(b) $\overline{y} = C_1 e^{x\sqrt{\frac{p}{2}}} + C_2 e^{-x\sqrt{\frac{p}{2}}}$
(c) $\overline{y} = C_1 e^{px} + C_2 e^{-px}$
(d) None of these

24. Laplace transform of $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, x > 0, t > 0$, $\overline{y} - L\{y\}$, $y(x, 0) = 0$ is :
(a) $(D^2 + 1)\,\overline{y} = 0$ (b) $(D^2 - 1)\overline{y} = 0$
(c) $(D^2 - p)\overline{y} = 0$ (d) $(D^2 + p)\,\overline{y} = 0$

25. The solution of $\int_0^t \frac{F(u)\,du}{\sqrt{t-u}} = 1$ is :
(a) $\sqrt{\frac{\pi}{t}}$ (b) $\sqrt{\frac{t}{\pi}}$ (c) $\frac{1}{\pi\sqrt{t}}$ (d) $\sqrt{\pi t}$

26. The Laplace transform of $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$ with $y = 0 = \frac{\partial y}{\partial t}$ at $t = 0$ is :
(a) $L\{y\} = C_1 e^{px} + C_2 e^{-px}$
(b) $L\{y\} = C_1 e^{pt} + C_2 e^{-pt}$
(c) $L\{y\} = C_1 e^{\frac{p}{2}x} + C_2 e^{-\frac{p}{2}x}$
(d) None of these

27. The solution of $F(t) = 1 + \int_0^t f(u)\sin(t-u)\,du$ is :
(a) $1 + \frac{t}{2}$ (b) $2 + \frac{t^2}{2}$ (c) $2 - \frac{t^2}{2}$ (d) $1 + \frac{t^2}{2}$

28. The solution of $F'(t) = \sin t + \int_0^t F(t-u)\cos u\,du$ with $F(0) = 0$ is :
(a) 0 (b) t (c) $\frac{t}{2}$ (d) $\frac{t^2}{2}$

29. The solution of $\int_0^t F(u)F(t-u)du = t$ is :
(a) 0 (b) $\pm t$ (c) $\pm\frac{t^2}{2}$ (d) ± 1

30. The solution of $\int_0^t \frac{f(u)\,du}{(t-u)^{1/3}} = t$ is :
(a) $\frac{\sqrt{3}\,t^{1/3}}{2\pi}$ (b) $\frac{3\sqrt{3}\,t^{1/2}}{\pi}$
(c) $\frac{3\sqrt{3}\,t^{1/3}}{2\pi}$ (d) $\frac{3t^{1/2}}{2\pi}$

31. The transformed differential equation after Laplace transform of $y''t + y' - y = 0$ if $y(0) = 0, y'(0) = 1$ is :
(a) $\frac{d\overline{y}}{dp} - \left(p - \frac{2}{p}\right)\overline{y} = -\frac{1}{p}$ (b) $\frac{d\overline{y}}{dp} - p\overline{y} = \frac{1}{p}$
(c) $\frac{d\overline{y}}{dp} - (p-2)\,\overline{y} = -\frac{1}{p}$ (d) $\frac{d\overline{y}}{dp} + p\overline{y} = -\frac{1}{p}$

32. After solving $(D-2)\,x - (D-2)\,y = \sin t$ and $(D^2 + 1)\,x + 2Dy = 0$, $x(0) = x'(0) = y(0) = 0$, $L\{x\}$ is:
(a) $\frac{2p}{(p^2+1)\,(p-1)}$ (b) $\frac{2p}{(p^2+1)\,(p^2-3p+2)}$
(c) $\frac{2p}{(p^2+1)\,(p^3-3p-2)}$ (d) None of these

33. Taking Laplace trasnform of $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 1 - e^{-t}$, $r < x < 1, t > 0, u(x, 0) = x$, new equation is :
(a) $(D - p)\overline{u} = \frac{1}{p(p+1)}$
(b) $(D + p)\,\overline{u} = \frac{1}{p(p-1)} + x$
(c) $(D + p)\overline{u} = \frac{1}{p+1} + x$
(d) $(D - p)\overline{u} = \frac{1}{p(p+1)} - x$

34. The solution of $(D^2 - 2D + 2)\,y = 0$, $y = Dy = 1$ at $t = 0$ is :
(a) $e^t + e^{-t}$ (b) $e^t\cos t$
(c) $e^t\sin t$ (d) $e^{-t}\sin t$

35. If $ty'' + 2y' + ty = 0$, $y(0) = 1$, $y(\pi) = 0$ then $L\{y\}$ is given by :
(a) $\tan p + A$ (b) $\tan^{-1} p + A$
(c) $-\tan p + A$ (d) $-\tan^{-1} p + A$

36. The differential of $\bar{y}=L\{y\}$ for $[tD^2+(t-1)\,D-1]\,y=0$ if $y(0)=5,\ y(\infty)=0$:

(a) $\dfrac{d\bar{y}}{dp}+p\bar{y}=\dfrac{10}{p}$

(b) $\dfrac{d\bar{y}}{dp}+\dfrac{3p}{p^2+1}\bar{y}=\dfrac{10}{p^2+p}$

(c) $\dfrac{d\bar{y}}{dp}+\dfrac{3p+2}{p^2+p}\bar{y}=\dfrac{10}{p^2+p}$

(d) None of these

37. The value of $L\{y\}$ for $(D^2+1)\,y=6\cos 2t$ if $y(0)=y'(0)=0$:

(a) $\dfrac{p}{p(p^2+1)}$ (b) $\dfrac{6p}{p(p^2+4)}$

(c) $\dfrac{6p}{(p^2+1)\,(p^2+4)}$ (d) None of these

38. If $(D^2+m^2)\,y=a\sin nt,\ t>0$ with $y(0)=y'(0)=0$ then $L\{y\}$ is :

(a) $\dfrac{an\,(p^2+n^2)}{p^2+m^2}$ (b) $\dfrac{n\,(p^2+n^2)}{p^2-m^2}$

(c) $\dfrac{a^n(p^2+m^2)}{p^2+n^2}$ (d) None of these

39. The solution $L\{y\}$ of $(D^2+9)\ y=18t$ if $y\ (0)=y'(0)=0$ is :

(a) $\dfrac{p^2}{p^2+9}$ (b) $\dfrac{18p^2}{p^2+9}$

(c) $\dfrac{18p}{p^2+9}$ (d) p^2+9

40. If $(D^2-4D+5)\ y=t^2$ with $y(0)=y'(0)=0$ then $L\{y\}$ is :

(a) $\dfrac{1}{p^2-4p+5}$ (b) $\dfrac{2}{p^3(p^2+1)}$

(c) $\dfrac{2}{p^3(p^2-4p+5)}$ (d) None of these

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(b)	**3.**	(c)	**4.**	(a)	**5.**	(c)	**6.**	(d)	**7.**	(a)	**8.**	(d)	**9.**	(a)	**10.**	(b)
11.	(c)	**12.**	(b)	**13.**	(a)	**14.**	(b)	**15.**	(d)	**16.**	(c)	**17.**	(d)	**18.**	(b)	**19.**	(d)	**20.**	(a)
21.	(d)	**22.**	(d)	**23.**	(b)	**24.**	(c)	**25.**	(c)	**26.**	(a)	**27.**	(d)	**28.**	(d)	**29.**	(d)	**30.**	(c)
31.	(a)	**32.**	(c)	**33.**	(d)	**34.**	(b)	**35.**	(d)	**36.**	(c)	**37.**	(c)	**38.**	(a)	**39.**	(b)	**40.**	(c)

HINTS AND SOLUTIONS

2. $(D+1)\ y=1,\ y\ (0)=2$

Taking Laplace transform

$$pL(y)-y(0)+L\{y\}=L\{1\}=\frac{1}{p}$$

$$p\bar{y}-2+\bar{y}=\frac{1}{p}$$

$$\Rightarrow\qquad (p+1)\bar{y}=2+\frac{1}{p}=\frac{2p+1}{p}$$

$$\bar{y}=\frac{2p+1}{p(p+1)}=\frac{1}{p+1}+\frac{1}{p}$$

Taking inverse Laplace

$$y=L^{-1}\left[\frac{1}{p+1}\right]+L^{-1}\left[\frac{1}{p}\right]=e^{-t}+1$$

3 $(D^2+1)\ y=0,\ y(0)=0,\ y'(0)=1$

Taking laplace transform we get

$$p^2L\{y\}-py(0)-y'(0)+L\{y\}=0$$

$$(p^2+1)\ L\{y\}=1$$

$$\Rightarrow\qquad L\{y\}=\frac{1}{p^2+1}$$

$$y=L^{-1}\left\{\frac{1}{p^2+1}\right\}=\sin t$$

5. Laplace transform of $D^3y=0$

$$p^3L\{y\}-p^2y(0)-py'(0)-y''(0)=0$$

Given $y(0)=y'(0)=0,\ y''(0)=1$, put these values in above equation we get

$$L\{y\} = \frac{1}{p^3}$$

⇒ $$y = L^{-1}\left\{\frac{1}{p^3}\right\} = \frac{t^2}{2!} = \frac{t^2}{2}$$

7. Given equation is $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$, taking the Laplace transform we have

$$L\left\{\frac{\partial y}{\partial x}\right\} = 2L\left[\frac{\partial y}{\partial t}\right] + L\{y\}$$

or $$\frac{d\bar{y}}{dx} = 2\{p\bar{y} - y(x, 0)\} + \bar{y}$$

∵ $y(x, 0) = 6e^{-3x}$

or $$\frac{d\bar{y}}{dx} - (2p + 1)\,\bar{y} = -12e^{-3x}$$

8. Taking the Laplace transform of

$(D^2 - D - 6)\, y = 2$ we get

$L\{y''\} - L\{y'\} - 6L\{y\} = L[2]$

$p^2L\{y\} - py(0) - y'(0) - [pL(y) - y\,(0)] - 6L\{y\} = \frac{2}{p}$

$p^2L\{y\} - p - pL\{y\} + 1 - 6L\{y\} = \frac{2}{p}$

$(p^2 - p - 6)L\{y\} = \frac{2}{p} + p - 1 = \frac{p^2 - p + 2}{p}$

∴ $$L\{y\} = \frac{p^2 - p + 2}{p(p^2 - p - 6)}$$

10. Taking Laplace transform of $3\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$ we have

$$3L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = L\left\{\frac{\partial y}{\partial t}\right\}$$

$$3\frac{d^2\bar{y}}{dx^2} = p\bar{y}\,(x, p) - y(x, 0)$$

$$\frac{d^2\bar{y}}{dx^2} - \frac{p}{3}\bar{y} = -10\cos 5x$$

12. Given equation is $F(t) = 1 + 2F(t) * e^{-2t}$

Taking Laplace transform of both sides we get

$$L\{F(t)\} = L\{1\} + 2L\{F(t) * e^{-2t}\} = \frac{1}{p}$$
$$+2\,[L\{F(t)\}.Le^{-2t}]$$

$$= \frac{1}{p} + \frac{2}{p+2}L\{F(t)\}$$

or $$L\{F(t)\} = \frac{p+2}{p^2}$$

∴ $$F(t) = L^{-1}\left\{\frac{1}{p}\right\} + 2L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$F(t) = 1 + 2t$$

17. Taking Laplace transform of given equation we have

$L\{y''\} + L\{y\} = L\{t\}$

$$p^2L\{y\} - py(0) - y'(0) + L\{y\} = \frac{1}{p^2}$$

put $y\,(0) = A$

$$(p^2 + 1)L\{y\} = Ap + \frac{(p^2 + 1)}{p}$$

$$L\{y\} = A\frac{p}{p^2 + 1} + \frac{1}{p^2} - y = A\cos t + t$$

But $y(\pi) = 0$

so $0 = A\cos\pi + \pi$

⇒ $A = \pi$

∴ $y = \pi\cos t + t$

20. $$F(t) = 4t - 3F(t) * \sin t$$

Taking Laplace transform of this equation

$$L\{F(t)\} = 4L\{t\} - 3L\{F(t) * \sin t\}$$
$$= 3L\{t\} - 3L\{F(t)\}.L\{\sin t\}$$
$$= \frac{4}{p^2} - 3L\{F(t)\}\frac{1}{p^2 + 1}$$

∴ $$L\{F(t)\} = \frac{4(p^2 + 1)}{p^2(p^2 + 4)}$$

$$F(t) = 4\left[L^{-1}\left(\frac{1}{p^2 - 4}\right) + L^{-1}\left(\frac{1}{p^2(p^2 + 4)}\right)\right]$$

$$F(t) = 4\left[\frac{\sin 2t}{2} + \frac{t}{4} - \frac{\sin 2t}{8}\right] = \frac{3}{2}\sin 2t + t$$

23. Taking Laplace transform of $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$

$$L\left\{\frac{\partial y}{\partial t}\right\} = 2L\left\{\frac{\partial^2 y}{\partial x^2}\right\} \text{ or } p\bar{y} - y(x, 0) = 2\frac{d^2\bar{y}}{dx^2}$$

or $$\frac{d^2\bar{y}}{dx^2} - \frac{p}{2}y^2 = 0$$

∵ $\bar{y}(x, 0) = 0$

so its general solution is

$$\bar{y} = C_1e^{x\sqrt{p/2}} + C_2e^{-x\sqrt{p/2}}$$

25. Given $F(t) * t^{-1/2} = 1$, taking Laplace transform

$$L\{F(t)\}.L\{t^{-1/2}\} = L\{1\}$$

⇒ $$L\{F(t)\}\frac{\sqrt{\pi}}{p^{1/2}} = \frac{1}{p}$$

$$L\{F(t)\} = \frac{1}{\sqrt{\pi}}\frac{1}{p^{1/2}}$$

$$\therefore \qquad F(t) = \frac{1}{\sqrt{\pi}} L^{-1}\left[\frac{1}{p^{1/2}}\right]$$

$$F(t) = \frac{1}{\pi\sqrt{t}}$$

28. Given equation is $F'(t) = \sin t + F(t) * \cos t$

Taking Laplace transform we have

$$L\{F'(t)\} = L\{\sin t\} + L\{F(t) * \cos t\}$$

$$pL\{F(t)\} - F(0) = \frac{1}{p^2+1} + L\{F(t)\}.L\{\cos t\}$$

$$pL\{F(t)\} = \frac{1}{p^3+1} + L\{F(t)\}.\frac{p}{p^2+1}$$

$$L\{F(t)\} = \frac{1}{p^3}$$

$$\Rightarrow \qquad F(t) = L^{-1}\left\{\frac{1}{p^3}\right\} = \frac{t^2}{2}$$

31. Taking laplace transform of $y'' + ty' - y = 0$

$$L\{y''\} + L\{ty'\} - L\{y\} = 0$$

$$p^2\bar{y} - py\,(0) - y'(0) - \frac{d}{dp}\{L(y')\} - \bar{y} = 0$$

$$(p^2-1)\bar{y} - 1 - \frac{d}{dp}\{p\bar{y} - y\,(0)\} = 0$$

$$\because \qquad y'(0) = 1$$

$$\text{or} \qquad (p^2-1)\bar{y} - 1 - \left\{\bar{y} + p\frac{d\bar{y}}{dp}\right\} = 0$$

$$\because \qquad y\,(0) = 0$$

$$\text{or} \qquad \frac{d\bar{y}}{dp} - \left(p - \frac{p}{2}\right)\bar{y} = -\frac{1}{p}$$

33. Taking Laplace transform of $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t} = 1 - e^{-t}$

we have $L\left\{\dfrac{\partial u}{\partial x}\right\} - L\left\{\dfrac{\partial u}{\partial t}\right\} = L\{1 - e^{-t}\}$

$$\frac{d\bar{u}}{dx} - \{p\bar{u} - u\,(x, 0)\} = \frac{1}{p} - \frac{1}{p+1}$$

$$\therefore \qquad u\,(x, 0) = x$$

$$\frac{d\bar{u}}{dx} - p\bar{u} + x = \frac{1}{p(p+1)}$$

$$\frac{d\bar{u}}{dx} - p\bar{u} = \frac{1}{p(p+1)} - x$$

35. Taking Laplace transform of $ty'' + 2y' + ty = 0$

$$L\{ty''\} + 2L\{y'\} + L\{ty\} = 0$$

$$\text{or} \quad -\frac{d}{dp}L\,\{y''\} + 2L\,\{y'\} - \frac{d}{dp}L\,\{y\} = 0$$

$$\text{or} \quad -\frac{d}{dp}\{p^2\bar{y} - py(0) - y'(0)\} + 2\{p\bar{y}$$

$$-y(0)\} - \frac{d\bar{y}}{dp} = 0 - (p^2+1)\frac{d\bar{y}}{dp} - 1 = 0$$

$$\text{or} \qquad \frac{d\bar{y}}{dp} = -\frac{1}{p^2+1}$$

$$\text{so} \qquad \bar{y} = A - \tan^{-1} p$$

37. Taking Laplace transform of $(D^2+1)y = 6\cos 2t$ we have

$$L\{y''\} + L\{y\} = 6L\{\cos 2t\}$$

$$p^2L\{y\} - py(0) - y'(0) + L\{y\} = \frac{6p}{p^2+4}$$

$$\therefore \qquad y(0) = y'(0) = 0 \text{ so}$$

$$(p^2+1)\;L\{y\} = \frac{6p}{p^2+4}$$

$$\Rightarrow \qquad L\{y\} = \frac{6p}{(p^2+1)\;(p^2+4)}$$

39. Taking Laplace transform of $(D^2+9)\;y = 18t$ we have

$$L\{y''\} + 9L\{y\} = 18L\{t\}$$

$$p^2L\{y\} - py(0) - y'(0) + 9L\{y\} = \frac{18}{p^2}$$

$$(p^2+9)L\{y\} = \frac{18}{p^2}$$

$$\therefore \qquad y(0) = y(0) = 0$$

$$\text{So,} \qquad L\{y\} = \frac{18}{p^2(p^2+9)}$$

40. Taking Laplace transform of given equation we have

$$L\{y''\} - 4L\{y\} + 5L\{y\} = L\{t^2\}$$

$$p^2L\{y\} - py(0) - y'(0) - 4\{pL(y)$$

$$-y\,(0)\} + 5L\{y\} = \frac{2}{p^3}$$

$$(p^2 - 4p + 5)\,L\{y\} = \frac{2}{p^3}$$

$$\therefore \qquad y(0) = y'(0) = 0$$

$$\therefore \qquad L(y) = \frac{2}{p^3(p^2 - 4p + 5)}$$

* * *

CHAPTER 17

Fourier Transforms

FOURIER TRANSFORM

1. Let $f(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$, then

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$$

is called the Fourier transforms of $f(x)$ and denoted by $F\{f(x)\}$ or $\hat{f}(p)$

The function $f(x)$ is called the inverse Fourier transform of $\hat{f}(p)$ *i.e.*

$$f(x) = F^{-1}\{\hat{f}(p)\}$$

2. If $\hat{f}(p)$ is the Fourier transform of $f(x)$ and if $f(x)$ satisfies the Dirichlet condition in very finite interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)\, dx|$ is convergent, then at every point of continuity of $f(x)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{-ipx} dp$$

Some authors also define the Fourier and inverse Fourier transform as follows

(i) $\hat{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$

and $f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \hat{f}(p)\, dp$

(ii) $\hat{f}(p) = \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$

and $f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \hat{f}(p)\, dp$

(iii) $\hat{f}(p) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$

and $f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx} dp$

3. The infinite Fourier sine transform of $f(x)$, $r < x < \infty$ is denoted by $F_s\{f(x)\}$ or $\hat{f}_s(p)$ and is defined as

$$F_s\{f(x)\} = \hat{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin px\, dx$$

The function $f(x)$ is called the inverse Fourier sine transform of $\hat{f}_s(p)$ *i.e.*

$$f(x) = F_s^{-1}\{\hat{f}_s(p)\}$$

Some authors define it as

$$\hat{f}_s(p) = \int_0^{\infty} f(x) \sin\, px\, dx$$

This is called Fourier sine transform.

4. If $\hat{f}_s(p)$ is the Fourier sine transform of the function $f(x)$ which satisfies the Dirichlet's conditions in $(0, e)$ and is such that $\int_0^{\infty} 1 f(x)\, dx$ exists, then

$f(x) = \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} \hat{f}_s(p) \sin px\, dx$ at every point of continuity of $f(x)$

This is called inverse formula for Fourier sine transform.

5. The infinite Fourier cosine transform of $f(x)$, $0 < x < \infty$ is denoted by $F_c\{f(x)\}$ or $\hat{f}_c(p)$ and is defined as

$$F_c\{f(x)\} = \hat{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px\, dx$$

The function $f(x)$ is called the inverse Fourier cosine transform of $\hat{f}_c(p)$ *i.e.*,

$$f(x) = F_c^{-1}\{\hat{f}_c(p)\}$$

Some authors define $\hat{f}_c(p) = \int_0^{\infty} f(x) \cos px\, dx$

This is called Fourier cosine transform.

6. If $\hat{f}_c(p)$ is the Fourier cosine transform of the function $f(x)$ which satisfies the Dirichlet conditions in every finite interval $(0, l)$ and is such that $\int_0^\infty |f(x)|\,dx$ exists, then

$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \hat{f}_c(p)\cos px\,dp$ at every point of continuity of $f(x)$.

This is called inversion formula for infinite Fourier cosine transform.

PROPERTIES OF FOURIER TRANSFORM

1. **Linearity Property**

If $\hat{f}(p)$ and $\hat{g}(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{af(x) + by(x)\} = a\hat{f}(p) + b\hat{g}(p)$$

2. **Change of Scale Property**

(i) If $\hat{f}(p)$ is the complex Fourier transform of $f(x)$, the complex Fourier transform of $f(ax)$ is $\frac{1}{a}\hat{f}\left(\frac{p}{a}\right)$.

(ii) If $\hat{f}_s(p)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $f(ax)$ is

$$\frac{1}{a}\hat{f}_s\left(\frac{p}{a}\right)$$

(iii) If $\hat{f}_c(p)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $f(ax)$ is $\frac{1}{a}\hat{f}_c\left(\frac{p}{a}\right)$.

3. **Shifting Property**

If $\hat{f}(p)$ is the complex Fourier transform of $f(x)$, then the complex Fourier transform of $f(x-a)$ is $e^{-ipa}\hat{f}(p)$.

4. **Modulation Theorem**

If$\hat{f}(p)$ is the complex Fourier transform of $f(x)$ then the Fourier transform of $f(x)\cos ax$ is

$$\frac{1}{2}[\hat{f}(p-a) + \hat{f}(p+a)]$$

CONVOLUTION

1. The function

$$F * G = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)\,G(x-u)\,du$$

is called the convolution or Falting of two integrable functions F andf G over $(-\infty, \infty)$.

Some define it as

$$F * G = \int_{-\infty}^{\infty} F(u), G(x-u)\,du$$

2. If $F\{f(x)\}$ and $F\{g(x)\}$ are the Fourier transforms of the functions $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms *i.e.*,

$$F\{f(x) * g(x)\} = F\{f(x)\}.F\{g(x)\}$$

This is called convolution or Falting Theorem.

FOURIER TRANSFORM OF THE DERIVATIVES

1. The Fourier transform of $f'(x)$, the derivative of $f(x)$ is $-ip\hat{f}(p)$, where $\hat{f}(p)$ is the Fourier transform of$f(x)$.

2. The Fourier transform of $f^n(x)$, the n^{th} derivative of $f(x)$ is $(-ip)^n$ times the Fourier transform of $f(x)$ provided that the first $(n-1)$ derivatives of $f(x)$ Vanish as $x \to \pm\infty$.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The infinite Fourier cosine transform of $f(x)$ is $\hat{f}_c(p)$ is :

(a) $\sqrt{\frac{2}{\pi}}\int_0^x f(x)\cos px\,dx$ (b) $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx$

(c) $\sqrt{\frac{2}{\pi}}\int_0^\infty \cos px\,dx$ (d) None of these

2. If $\hat{f}(p)$ and $\hat{g}(p)$ are Fourier transform of $f(x)$ and $g(x)$, then $F\{af(x)+bg(x)\}$ is :

(a) $\hat{f}(ap)+\hat{g}(bp)$ (b) $a\hat{f}(p)+\hat{g}(bp)$

(c) $\hat{f}(ap)+b\hat{g}(p)$ (d) $a\hat{f}(p)+b\hat{g}(p)$

3. If $F\{f(x)\}=\hat{f}(p)$ then $F\{f(ax)\}$ is :

(a) $\hat{f}\left(\frac{p}{a}\right)$ (b) $\frac{1}{a}\hat{f}\left(\frac{p}{a}\right)$

(c) $a\hat{f}\left(\frac{p}{a}\right)$ (d) $\frac{1}{a}\hat{f}(ap)$

4. If $F(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ then for $p\neq 0$, $F\{f(x)\}$ is :

(a) $\frac{\sin pa}{p\sqrt{2\pi}}$ (b) $\frac{\sin pa}{a\sqrt{2\pi}}$ (c) $\frac{2\sin pa}{p\sqrt{2\pi}}$ (d) $\frac{2\sin pa}{a\sqrt{2\pi}}$

5. The values of $F_s\{e^{-x}\}$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{1}{p^2+1}$ (b) $\sqrt{\frac{2}{\pi}}\frac{1}{p^2-1}$

(c) $\sqrt{\frac{2}{\pi}}\frac{p}{p^2-1}$ (d) $\sqrt{\frac{2}{\pi}}\frac{p}{p^2+1}$

6. If $F_C\left\{\frac{1}{1+x^2}\right\}=\sqrt{\frac{\pi}{2}}e^{-p}$ then $F_S\left\{\frac{x}{1+x^2}\right\}$ is :

(a) 0 (b) e^{-p} (c) $\sqrt{\frac{2}{\pi}}e^{-p}$ (d) $\sqrt{\frac{\pi}{2}}e^{-p}$

7. If $\hat{f}_c(p)=\pi e^{-p/2}$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{1}{x}$ (b) $\sqrt{\frac{\pi}{2}}\frac{1}{1+x^2}$

(c) $\sqrt{\frac{2}{\pi}}\frac{e^{-x}}{2}$ (d) None of these

8 If $F\{f(x)\}=\hat{f}(p)$ then $F\{f^n(x)\}$ is :

(a) $-\hat{f}^n(p)$ (b) $(-p)^n\hat{f}(p)$

(c) $(-ip)^n\hat{f}(p)$ (d) None of these

9. The infinite Fourier sine transform of $f(x)$ *i.e.*, $\hat{f}_s(p)$ is :

(a) $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dp$ (b) $\sqrt{\frac{2}{\pi}}\int_0^x f(x)\sin px\,dx$

(c) $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx$ (d) $\sqrt{\frac{2}{\pi}}\int_0^x f(x)\sin px\,dp$

10. If $F_c\{f(x)\}=\hat{f}_c(p)$ then $F_C\{f(ax)\}$ is :

(a) $\frac{1}{a}\hat{f}_c\left(\frac{p}{a}\right)$ (b) $a\hat{f}_c\left(\frac{p}{a}\right)$

(c) $\frac{1}{a}\hat{f}_c\left(\frac{a}{p}\right)$ (d) $\frac{1}{a}\hat{f}_c(ap)$

11. If $F\{f(x)\}=\hat{f}(p)$ then$F\{f(x-a)\}$ is :

(a) $e^{-ipa}\hat{f}(p)$ (b) $e^{-ip}\hat{f}\left(\frac{p}{a}\right)$

(c) $e^{ipa}\hat{f}(p)$ (d) $e^{ipa}\hat{f}\left(\frac{p}{a}\right)$

12. $\frac{1}{2}\{\hat{f}_s(p+a)-\hat{f}_s(p-a)\}$ is equal to :

(a) $F_S\{f(x)\cos ax\}$ (b) $F_S\{f(x)\sin ax\}$

(c) $F_C\{f(x)\sin ax\}$ (d) $F_C\{f(x)\cos ax\}$

13. If $f(x)=\begin{cases}\frac{\sqrt{2\pi}}{2a}, & |x|\le a\\ 0, & |x|>a\end{cases}$ then $F\{f(x)\}$ is :

(a) $\frac{\cos ap}{ap}$ (b) $\frac{\sin ap}{ap}$

(c) $\frac{\sin p}{ap}$ (d) $\frac{\cos p}{p}$

14. If$f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x>a\end{cases}$ then $F_C\{f(x)\}$ is :

(a) $\frac{1}{\sqrt{2\pi}}\left\{\frac{\sin ap}{p}+\frac{\cos ap}{p}\right\}$

(b) $\frac{1}{\sqrt{2\pi}}\left\{\frac{\cos(1+p)a}{1+p}+\frac{\cos(1-p)a}{1-p}\right\}$

(c) $\frac{1}{\sqrt{2\pi}}\left\{\frac{\sin(1+p)a}{1+p}+\frac{\cos(1-p)a}{1-p}\right\}$

(d) $\frac{1}{\sqrt{2\pi}}\left\{\frac{\sin(1+p)a}{1+p}+\frac{\sin(1-p)a}{1-p}\right\}$

15. If $f(x)=\begin{cases}0 & 0<x<a\\ x & a\le x\le b\\ 0 & x>b\end{cases}$ then $\hat{f}_s(p)$ is :

(a) $\sqrt{\frac{2}{\pi}}\int_0^\infty x\sin px\,dx$ (b) $\sqrt{\frac{2}{\pi}}\int_0^\infty \sin px\,dx$

(c) $\sqrt{\frac{2}{\pi}}\int_a^b x\sin px\,dx$ (d) $\sqrt{\frac{2}{\pi}}\int_a^b \sin px\,dx$

16. If $\hat{f}_s(p) = \frac{\pi}{2}$ then $f(x)$ is :

(a) $\sqrt{\frac{\pi}{2}}x$ (b) $\sqrt{\frac{\pi}{2}}$

(c) $\sqrt{\frac{\pi}{2}}x^2$ (d) $\sqrt{\frac{\pi}{2}}\frac{1}{x}$

17. If $\hat{f}_c(p) = e^{-p}$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}(x^2+1)$ (b) $\sqrt{\frac{2}{\pi}}x$

(c) $\sqrt{\frac{2}{\pi}}\frac{x}{(x^2+1)}$ (d) $\sqrt{\frac{2}{\pi}}\frac{1}{(x^2+1)}$

18. If $\hat{f}_c(p)$ is the Fourier cosine transform of $f(x)$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(p)\cos px\,dx$ (b) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(p)\,dp$

(c) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(p)\cos px\,dp$ (d) None of these

19. If $F_s\{f(x)\} = \hat{f}_s(p)$ then $F_s\{f(ax)\}$ is :

(a) $a\hat{f}_s\left(\frac{p}{a}\right)$ (b) $\frac{1}{a}\hat{f}_s(ap)$

(c) $\frac{1}{a}\hat{f}_s\left(\frac{p}{a}\right)$ (d) $a\hat{f}_s(ap)$

20. If $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ then for $p = 0$, $\hat{f}(p)$ is :

(a) $2a$ (b) $\frac{a}{\sqrt{2\pi}}$ (c) $\frac{2a^2}{\sqrt{2\pi}}$ (d) $\frac{2a}{\sqrt{2\pi}}$

21. $F_c\{e^{-x}\}$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{1}{1+p^2}$ (b) $\sqrt{\frac{2}{\pi}}\frac{p}{p^2+1}$

(c) $\sqrt{\frac{2}{\pi}}\frac{1}{p^2-1}$ (d) $\sqrt{\frac{2}{\pi}}\frac{p}{p^2-1}$

22. If $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ then Fourier transform of $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}(\cos ap - \sin ap)$ (b) $\sqrt{\frac{2}{\pi}}(p\cos ap + \sin ap)$

(c) $-\frac{i}{p^2}\sqrt{\frac{2}{\pi}}(ap\cos ap - \sin ap)$

(d) None of these

23 If $\hat{f}_c(p) = \frac{\pi}{2}$ then $f(x)$ is :

(a) 0 (b) $\sqrt{\frac{\pi}{2}}x$ (c) $\sqrt{\frac{\pi}{2}}$ (d) $\sqrt{\frac{\pi}{2}}.\frac{x}{2}$

24. If $\hat{f}_c(p) = \begin{cases} \frac{\left(a - \frac{p}{2}\right)}{\sqrt{2\pi}}, & p < 2a \\ 0, & p \ge 2a \end{cases}$ then $f(x)$ is :

(a) $\frac{\sin^2 ax}{\pi}$ (b) $\frac{\cos^2 ax}{\pi x^2}$

(c) $\frac{\cos^2 ax}{\pi x}$ (d) $\frac{\sin^2 ax}{\pi x^2}$

25. If $\hat{f}_s(p) = \frac{e^{-ap}}{p}$ then $f(x)$ is :

(a) $\sqrt{\frac{\pi}{2}}\tan^{-1}(x)$ (b) $\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$

(c) $\sqrt{\frac{2}{\pi}}\sin^{-1}\left(\frac{x}{a}\right)$ (d) $\sqrt{\frac{2}{\pi}}\frac{e^{-ax}}{x}$

26. If $\hat{f}_s(p)$ is the Fourier sine transform of $f(x)$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(p)\sin px\,dp$ (b) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(p)\sin px\,dp$

(c) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(p)\,dp$ (d) $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(p)\,dx$

27. If $F\{f(x)\} = \hat{f}(p)$ then $F\{f(x)\cos ax]$ is :

(a) $\hat{f}(p)\cos ap$ (b) $\hat{f}(p)\sin ap$

(c) $\hat{f}(p)\left(\frac{\cos ap + \sin ap}{2}\right)$ (d) $\frac{\hat{f}(p-a) + \hat{f}(p+a)}{2}$

28. If F and G are two integrable functions over $(-\infty, \infty)$ then the convolution $F * G$ is :

(a) $\frac{1}{\sqrt{2\pi}}\int_0^\infty F(u)\,G(x-u)\,du$

(b) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(u)\,G(u)\,du$

(c) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(u)\,G(x-u)\,du$

(d) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(u)\,G(x+u)\,du$

29. If $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$ then $F_c\{f(x)\}$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{\cos p}{p^2}$ (b) $\sqrt{\frac{2}{\pi}}\frac{\sin p}{p^2}$

(c) $2\sqrt{\frac{2}{\pi}}\frac{\sin p}{p^2}(1-\cos p)$ (d) $2\sqrt{\frac{2}{\pi}}\sin p\,(1-\cos p)$

30. If $\hat{f}_s(p) = e^{-ap}$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2}$ (b) $\sqrt{\frac{2}{\pi}}(a^2+x^2)$

(c) $\sqrt{\frac{2}{\pi}}\frac{a^2+x^2}{x}$ (d) None of these

31. If $\hat{f}_s(p) = \frac{1}{p}$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}p$ (b) $\sqrt{\frac{\pi}{2}}p$

(c) $\sqrt{\frac{\pi}{2}}$ (d) $\frac{1}{p}\sqrt{\frac{2}{\pi}}$

32. If $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ then $F\{f(x)\}$ is :

(a) $\frac{2}{p}\sin ap$ (b) $\frac{p}{2}\sin ap$

(c) $\frac{2}{p}\cos ap$ (d) $\frac{p}{2}\cos ap$

33. If $f(x) = e^{-x}$ then $\lim\limits_{p\to\infty} f_s\{f(x)\}$ is :

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

34. The convolution of $f(x)$ and $g(x)$ is $f(x) * g(x)$ then $F\{f(x) * g(x)\}$ is :

(a) $F\{f(x)\} + F\{g(x)\}$ (b) $F\{f(x)\} - F\{g(x)\}$

(c) $F\left\{\frac{f(x)}{g(x)}\right\}$ (d) $F\{f(x)\} - F\{g(x)\}$

35. If $F\{f(x)\} = F(\hat{p})$ then $F\{f'(x)\}$ is :

(a) $\hat{f}'(p)$ (b) $ip\hat{f}(p)$

(c) $-p\hat{f}(p)$ (d) $-ip\hat{f}(p)$

36. If $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$ then find $F_s\{f(x)\}$:

(a) $2\sqrt{\frac{2}{\pi}}\frac{\sin p}{p^2}(1-\cos p)$ (b) $\sqrt{\frac{2}{\pi}}\frac{\cos p}{p^2}$

(c) $\sqrt{\frac{2}{\pi}}\cos p(1-\cos p)$ (d) $2\sqrt{\frac{2}{\pi}}\frac{\sin p}{p^2}(1-\sin p)$

37. If $\hat{f}_c(p) = e^{-ap}$ then $f(x)$ is :

(a) $\sqrt{\frac{2}{\pi}}\frac{a^2+x^2}{a}$ (b) $\sqrt{\frac{2}{\pi}}\frac{a}{x^2+a^2}$

(c) $\sqrt{\frac{2}{\pi}}a$ (d) $\sqrt{\frac{\pi}{2}}a$

38. If $F\{f(x)\}s = \hat{f}(p)$ then $F\{f(x)\cos ax\}$ is :

(a) $\frac{\hat{f}(p-a+\hat{f}(p+a)}{2}$ (b) $\frac{1}{2}\hat{f}(p-a)$

(c) $\frac{1}{2}\hat{f}(p+a)$ (d) $\frac{\hat{f}(p-a)-\hat{f}(p+a)}{2}$

39. $F_S[e^{-5x}]$ is equal to :

(a) $\frac{p^2+1}{5}$ (b) $\frac{p^2-1}{25}$

(c) $\frac{p(p-1)}{5}$ (d) $\frac{p}{p^2+25}$

40. $F_c\{e^{-2x}\}$ is equal to :

(a) $\sqrt{\frac{2}{\pi}}\left(\frac{1}{p^2+4}\right)$ (b) $\sqrt{\frac{2}{\pi}}\left(\frac{4}{p^2+4}\right)$

(c) $\sqrt{\frac{2}{\pi}}\left(\frac{2}{p^2+4}\right)$ (d) $\frac{4}{p^2-4}\left(\frac{2}{\pi}\right)$

41. If $f(x) = \frac{1}{x}$ then $F_s\{f(x)\}$ is :

(a) $p\sqrt{\frac{\pi}{2}}$ (b) $\frac{1}{p}\sqrt{\frac{\pi}{2}}$ (c) $\frac{1}{p}$ (d) $\sqrt{\frac{\pi}{2}}$

42. $F_s\{f(x)\}$ is equal to :

(a) $-\frac{d}{dp}F_c\{f(x)\}$ (b) $-\frac{d}{dx}F_c\{f(x)\}$

(c) $-\frac{d}{dp}F_s\{f(x)\}$ (d) None of these

43. $F_c\{xf(x)\}$ is equal to :

(a) $xf'(x)$ (b) $\frac{d}{dp}F_c\{f(x)\}$

(c) $-xf'(x)$ (d) $\frac{d}{dp}F_s\{f(x)\}$

44. If $F_s\{f(x)\} = (2\pi p)^{1/}$ then $f(x)$ is :

(a) $\frac{1}{x}$ (b) $\frac{1}{\sqrt{x}}$

(c) $x\sqrt{x}$ (d) $\frac{1}{x\sqrt{x}}$

45. If $\int_0^\infty f(x)\cos px\,dx = e^{-p}$ then $f(x)$ is :

(a) $\frac{1}{1+x^2}$ (b) $\frac{2}{\pi}\frac{1}{1+x^2}$

(c) $\sqrt{\frac{2}{\pi}}\frac{1}{1+x^2}$ (d) $\sqrt{\frac{2}{\pi}}\frac{1}{x}$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(d)	**3.**	(b)	**4.**	(c)	**5.**	(d)	**6.**	(d)	**7.**	(b)	**8.**	(c)	**9.**	(c)	**10.**	(a)
11.	(c)	**12.**	(c)	**13.**	(b)	**14.**	(b)	**15.**	(b)	**16.**	(d)	**17.**	(d)	**18.**	(c)	**19.**	(c)	**20.**	(d)
21.	(a)	**22.**	(c)	**23.**	(a)	**24.**	(d)	**25.**	(b)	**26.**	(a)	**27.**	(d)	**28.**	(c)	**29.**	(c)	**30.**	(a)
31.	(c)	**32.**	(a)	**33.**	(a)	**34.**	(d)	**35.**	(d)	**36.**	(a)	**37.**	(d)	**38.**	(a)	**39.**	(d)	**40.**	(c)
41.	(d)	**42.**	(a)	**43.**	(d)	**44.**	(d)	**45.**	(b)										

HINTS AND SOLUTIONS

4. Given that $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

Since $F\{f(x)\} = \hat{f}(p) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ipx}\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}\right)$$

$$= \frac{2\,i\sin pa}{ip\sqrt{2\pi}} = \frac{2\sin pa}{p\sqrt{2\pi}}$$

6. Given that $\hat{f}_c(p) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}\frac{\cos px}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}e^{-p}$

Differentiating both sides w.r.t. p we get

$$-\int_0^{\infty}\frac{x\sin px}{1+x^2}\,dx = -\frac{\pi}{2}e^{-p}$$

$$\therefore\ \hat{f}_s(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x}{1+x^2}\sin px\,dx = \sqrt{\frac{\pi}{2}}e^{-p}$$

7. Here $\hat{f}_c(p) = \dfrac{\pi e^{-p}}{2}$, so by Fourier cosine inversion formula we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_c(p)\cos px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\pi e^{-p}}{2}\cos px\,dp$$

$$= \sqrt{\frac{\pi}{2}}\left[\frac{e^{-p}}{1+x^2}(-\cos px + \sin px)\right]_0^{\infty} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{1+x^2}$$

13. Here, $f(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2a} & |x| \le a \\ 0 & |x| > a \end{cases}$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\frac{\sqrt{2\pi}}{2a}\,dx$$

$$= \frac{1}{2a}\left[\frac{e^{ipx}}{ip}\right]_{-a}^{a} = \frac{e^{ipa} - e^{-ipa}}{2ipa} = \frac{\sin pa}{pa}$$

15. $\hat{f}_s(p) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin px\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^{a} 0\sin px\,dx + \int_a^{b} x\sin px\,dx + \int_b^{\infty} 0\sin px\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_a^{b} x\sin px\,dx$$

16. By Fourier sine inverse formula we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_s(p)\sin px\,dp = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\pi}{2}\sin px\,dp$$

$$= \sqrt{\frac{\pi}{2}}\int_0^{\infty}\sin px\,dx$$

Since $\displaystyle\int_0^{\infty} e^{-ipx}\,dx = \left[\frac{e^{-ipx}}{-ix}\right]_0^{\infty} = \frac{1}{ix} = -\frac{i}{x}$

i.e. $\displaystyle\int_0^{\infty}(\cos px - i\sin px)\,dx = -\frac{i}{x}$

$\Rightarrow$ $\displaystyle\int_0^{\infty}\sin px\,dx = \frac{1}{x}$

$\therefore$ $f(x) = \sqrt{\dfrac{\pi}{2}}\left(\dfrac{1}{x}\right)$

20. $\hat{f}(p) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{ipx} f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\,dx$

$\therefore\ p = 0$ so $\hat{f}(D) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-a}^{a} dx = \frac{2a}{\sqrt{2\pi}}$

21. $\hat{f}_c(p) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\cos px\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos px\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+p^2}(-\cos px + p\sin px)\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+p^2}\right)$$

24. By Fourier cosine inversion formula, we have

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_c(p)\cos px\,dp$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{2a}\frac{1}{\sqrt{2\pi}}\left(a-\frac{p}{2}\right)\cos px\,dp$$

$$+\sqrt{\frac{2}{\pi}}\int_{2\alpha}^{\infty}0.\cos px\,dp$$

$$=\left[\frac{1}{\pi x}\left(a-\frac{p}{2}\right)\sin px\right]_0^{2a}-\left(\frac{\cos px}{2\pi x^2}\right)_0^{2a}$$

$$=\frac{1-\cos 2ax}{2\pi x^2}=\frac{2\sin^2 ax}{2\pi x^2}=\pi^{-1}x^{-2}\sin^2 ax$$

25. By Fourier cosine inversion formula

$$F_s^{-1}\left\{\frac{e^{-ap}}{p}\right\}=f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ap}\sin px}{p}dp$$

Differentiating w.r.t. x $\dfrac{df}{dx}=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}e^{-ap}\cos px\,dp$

$$\frac{df}{dx}=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}$$

Integrating we get

$$f=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)+A$$

But when $x=0$, $f=0$

so $A=0$ *i.e.* $f(x)=\sqrt{\dfrac{2}{\pi}}\tan^{-1}\left(\dfrac{x}{a}\right)$

29. $\hat{f}_c(p)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}f(x)\cos px\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 x\cos px\,dx+\int_1^2(2-x)\cos px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(\frac{x}{p}\sin px+\frac{\cos px}{p^2}\right)_0^1\right.$$

$$\left.+\left(\frac{2-x}{p}\sin px-\frac{\cos px}{p^2}\right)_1^2\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{p^2}[2\cos p-1-\cos 2p]=2\sqrt{\frac{2}{\pi}}\frac{\cos p}{p^2}(1-\cos p)$$

32. $F\{f(x)\}=\displaystyle\int_{-\infty}^{\infty}e^{ipx}f(x)\,dx=\int_{-a}^{a}e^{ipx}1dx=\left(\frac{e^{ipx}}{ip}\right)_{-a}^{a}$

$$=\frac{2}{ip}\left\{\frac{e^{ipa}-e^{-ipa}}{2}\right\}=\frac{2\sin ap}{p}\text{ when }p\neq 0$$

33. $F_S\{e^{-x}\}=\hat{f}_S(p)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}e^{-x}\sin px\,dx$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+p^2}(-\sin px-p\cos px)\right]_0^{\infty}$$

$$=\frac{p}{1+p^2}\sqrt{\frac{2}{\pi}}$$

So $\displaystyle\lim_{p\to\infty}\hat{f}_S(p)=0$

36. $\hat{f}_S(p)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}f(x)\sin px\,dx=\sqrt{\frac{2}{\pi}}[\int_0^1 x\sin px\,]\,dx$

$$+\int_1^2(2-x)\sin px\,dx]$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(-\frac{x\cos px}{p}+\frac{1}{p^2}\sin px\right)_0^1\right.$$

$$\left.+\left[-\left(\frac{2-x}{p}\right)\cos px-\frac{1}{p^2}\sin px\right]_1^2\right.$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2}{p^2}\sin p-\frac{1}{p^2}\sin 2p\right]=2\sqrt{\frac{2}{\pi}}\frac{\sin p}{p^2}(1-\cos p)$$

38. $F_S\{f(x)\cos ax\}=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}f(x)\cos ax\sin px\,dx$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^{\infty}f(x)\{\sin\,(p+a)x+\sin(p-a)x\}dx$$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\,(p+a)x\,dx\right.$$

$$\left.+\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\,(p-a)x\,dx\right]$$

$$=\frac{1}{2}[\hat{f}_S(p+a)+\hat{f}_S(p-a)]$$

40. $F_C[e^{-2x}]=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}e^{-2x}\cos px\,dx$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2x}}{4+x^2}(-2\cos px+x\sin px)\right]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\frac{2}{4+p^2}$$

41. $F_S\left[\dfrac{1}{x}\right]=\hat{f}_S(p)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}\frac{\sin px}{x}dx$

put $px=t$, $pdx=dt$

$$\hat{f}_S(p)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin t}{t}dt=\sqrt{\frac{2}{t\pi}}\frac{\pi}{2}=\sqrt{\frac{\pi}{2}}$$

43. The Fourier sine transform of $f(x)$

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin px\,dx = \hat{f}_S(p)$$

Differentiating w.r.t. p we get $\frac{d}{dp}F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}$ $\int_0^{\infty} xf(x)\cos px\,dx$

or $$\frac{d}{dp}F_S\{f(x)\} = F_C\{xf(x)\}$$

45. Given $\int_0^{\infty} f(x)\cos px\,dx = e^{-p}$

or $$\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos px\,dx = \sqrt{\frac{2}{\pi}}e^{-p} = F_C\{f(x)\}$$

$$\therefore\ f(x) = F_C^{-1}\left[\sqrt{\frac{2}{\pi}}e^{-p}\right] = \sqrt{\frac{2}{\pi}}.\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-p}\cos px\,dp$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} e^{-p}\cos px\,dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-p}}{1+x^2}\{-\cos px + x\sin px\}\right]_0^{\infty}$$

$$= \frac{2}{\pi}\frac{1}{1+x^2}$$

* * *

Unit-V

CHAPTER

18

Finite Fourier Transform

FINITE FOURIER SINE TRANSFORM

1. The finite fourier sine transform of $f(x)$ in the finite range $(0, e)$ is defined as

$$F_S\{f(x)\} = \hat{f}_S(p) = \int_0^l f(x)\sin\frac{\pi px}{l}\,dx$$

where p is an integer.

In the finite range $0 < x < \pi$ it is defined as

$$F_S\{f(x)\} = \hat{f}_S(p) = \int_0^\pi f(x)\sin px\,dx$$

2. The finite fourier cosine transform of $f(x)$ in $(0, l)$ is defined as

$$F_C\{f(x)\} = \hat{f}_C(p) = \int_0^l f(x)\cos\frac{\pi px}{l}\,dx$$

If the range is $(0, \pi)$ then

$$F_C\{f(x)\} = \hat{f}_C(p) = \int_0^\pi f(x)\cos px\,dx$$

INVERSE FORMULA FOR FINITE FOURIER TRANSFORM

1. If $\hat{f}_S(p)$ is the finite fourier sine transform of $f(x)$ over $(0, l)$ then the inverse formula for sine transform is

$$f(x) = \frac{2}{l}\sum_{p=1}^{\alpha}\hat{f}_S(p)\sin\frac{\pi px}{l}$$

If interval is $(0, \pi)$ then it becomes

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\alpha}\hat{f}_S(p)\sin px$$

2. If $\hat{f}_C(p)$ is the finite fourier cosine transform of $f(x)$ over $(0, l)$ then the inversion formula for cosine transform is

$$f(x) = \frac{1}{l}\hat{f}_C(x) + \frac{2}{l}\sum_{p=1}^{\infty}\hat{f}_C(p)\cos\frac{\pi px}{l}$$

where $\hat{f}_C(0) = \int_0^l f(x)\,dx$

If the interval is $(0, \pi)$ then it becomes

$$f(x) = \frac{1}{\pi}\hat{f}_C(0) + \frac{2}{\pi}\sum_{p=1}^{\alpha}\hat{f}_C(p)\cos px$$

where $\hat{f}_C(0) = \int_0^\pi f(x)\,dx$

3. Let $f(x, y)$ be a function of x, y defined in $0 \le x \le \pi$ and $0 \le y \le \pi$ then

$$\hat{f}_S(p, q) = \int_0^\pi\int_0^\pi f(x,y)\sin px\sin qy\,dx\,dy$$

Similarly,

$$\hat{f}_C(p, q) = \int_0^\pi\int_0^\pi f(x, y)\cos px\cos qy\,dx\,dy$$

PROPERTIES OF FINITE FOURIER TRANSFORM

1. If $f(x)$ is continuous and $f'(x)$ is sectionally continuous then

(i) $F_S\{f'(x)\} = -pF_C\{f(x)\},\ p = 1, 2, 3, \ldots$

(ii) $F_C\{f'(x)\} = pF_S\{f(x)\} - f(0)$

$$+(-1)^p f(\pi),\ p = 0, 1, 2, \ldots$$

2. If $F(x)$ and $G(x)$ be two functions on the interval $(-2\pi, 2\pi)$ then

$F(x) * G(x) = \int_{-\pi}^{\pi} F(x - y)\,G(y)\,dy$ is called the convolution of $F(x)$ and $G(x)$.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. The finite fourier sine transform of 1 is :

(a) $\frac{1+(-1)^p}{p}$ (b) $\frac{1-(-1)^p}{p}$

(c) $\frac{1+p^2}{p}$ (d) $\frac{1-p^2}{p}$

2. The finite fourier cosine transform of x is :

(a) $\frac{1+(-1)^p}{p}$ (b) $\frac{1-(-1)^p}{p^2}$

(c) $\frac{-1-(-1)^p}{p^2}$ (d) $\frac{-1+(-1)^p}{p^2}$

3. The finite fourier sine transform of $\left(1-\frac{x}{\pi}\right)$ is :

(a) p (b) $\frac{1}{p}$ (c) $\frac{\pi p}{1-p}$ (d) $\frac{p}{1+p}$

4. If $\hat{f}_S(p)$ is the finite fourier sine transform of $f(x)$ is $(0, \pi)$ then $f(x)$ is :

(a) $\int_0^\alpha \hat{f}_S(p)\sin px$ (b) $\sum_1^\alpha \hat{f}_S(p)\sin px$

(c) $\frac{2}{\pi}\sum_0^\alpha \hat{f}_S(p)\sin px\,dx$ (d) $\frac{2}{\pi}\int_0^\alpha \hat{f}_S(p)\sin px\,dp$

5. The finite fourier cosine transform of $f(x)$ is :

(a) $\Sigma f(x)\cos\frac{\pi px}{l}$ (b) $\int_0^l f(x)\frac{\cos \pi px}{l}dx$

(c) $\int_0^l f(x)\cos\frac{px}{l}dx$ (d) $\Sigma f(x)\cos\frac{px}{l}$

6. If $f(x) = \begin{cases} 1, & r < x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x < \pi \end{cases}$ then $\hat{f}_C(0)$ is :

(a) $\frac{2}{p}\cos \pi p$ (b) $\frac{2}{p}\cos\left(\frac{\pi p}{2}\right)$

(c) $\frac{1}{p}$ (d) 0

7. The finite fourier cosine transform of $\left(1-\frac{x}{\pi}\right)^2$ when $p = 0$ is :

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{3}$

8. If $f(x) = x^2$, $0 < x < 4$ then $\hat{f}_C(0)$ is :

(a) 0 (b) 1 (c) $\frac{3}{64}$ (d) $\frac{64}{3}$

9. The finite fourier cosine transform of 1 is :

(a) p (b) $\frac{1+(-1)^p}{p}$

(c) $1+(-1)^p$ (d) 0

10. The finite fourier cosine transform of x when $p = 0$, is :

(a) π (b) $-\pi$ (c) π^2 (d) $\frac{\pi^2}{2}$

11. The finite fourier sin transform of $\frac{x}{4\pi}$ is :

(a) $\frac{(-1)^p}{4p}$ (b) $\frac{(-1)^{p-1}}{4p}$

(c) $\frac{(-1)^{p+1}}{4p}$ (d) $\frac{(-1)^p}{p}$

12. If $\hat{f}_C(p) = -\frac{\cos\left(2p\frac{\pi}{3}\right)}{(2p+1)^2}$ and $0 < x < 1$ then $f(x)$ is :

(a) $\sum_{p=1}^{\alpha}\frac{\cos\left(2p\frac{\pi}{3}\right)}{(2p+1)^2}\cos p\pi x$

(b) $\sum_{1}^{\alpha}\frac{\cos\left(2p\frac{\pi}{3}\right)}{(2p+1)^2}\sin p\pi x$

(c) $1+\sum_{1}^{\alpha}\frac{\cos\left(2p\frac{\pi}{3}\right)}{(2p+1)^2}\sin p\pi x$

(d) $1+2\sum_{p=1}^{\alpha}\frac{\cos\left(2p\frac{\pi}{3}\right)}{(2p+1)^2}\cos p\pi x$

13. If $f(x) = \sin mx$, m is +ve integer and $m = p$ then $\hat{f}_S(p)$ is :

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) 0

14. If $f(x) = 2x$, $0 < x < 4$ then $\hat{f}_S(p)$ is :

(a) $\frac{4}{p\pi}\cos p\pi$ (b) $-\frac{4}{p\pi}\cos p\pi$

(c) $\frac{32}{p\pi}\sin p\pi$ (d) $-\frac{32}{p}\cos p\pi$

15. If $f(x) = \sin nx$ and $(n-p)$ is given in $0 < x < \pi$ then $\hat{f}_C(p)$ is :

(a) $\frac{1}{n+p}$ (b) $\frac{1}{n-p}$

(c) $\frac{1}{n+p}+\frac{1}{n-p}$ (d) 0

16. If $\hat{f}_S(p) = \dfrac{1-\cos p\pi}{p^2\pi^2}, 0 < x < \pi$ then $f(x)$ is :

(a) $\sum\limits_{p=1}^{\alpha}\left(\dfrac{1-\cos p\pi}{p}\right)\sin px$

(b) $\dfrac{2}{\pi^3}\sum\limits_{p=1}^{\alpha}\left(\dfrac{1-\cos p\pi}{p^2}\right)\sin px$

(c) $\displaystyle\int_{\phi}^{\alpha}\frac{\cos p\pi}{p}\sin px\, dx$

(d) $\dfrac{2}{\pi^2}\displaystyle\int_{1}^{\alpha}\left(\frac{1+\cos p\pi}{p^2}\right)\sin px$

17. The finite fourier sine transform of x is :

(a) $\dfrac{(-1)^{p+1}}{p}$ (b) $\dfrac{(-1)^p}{p}$

(c) $\dfrac{\pi(-1)^p}{p}$ (d) $\dfrac{p(-1)^p}{4\pi p^2}$

18. The finite fourier cosine transform of $\left(1-\dfrac{x}{\pi}\right)$ is :

(a) $\dfrac{1-(-1)^p}{\pi p^2}$ (b) $\dfrac{-1-(-1)^p}{\pi p^2}$

(c) $\dfrac{1+(-1)^p}{4\pi p^2}$ (d) $\dfrac{-1-(-1)^p}{4\pi p^2}$

19. If $f(x) = \begin{cases} 1; & 0 < x < \dfrac{\pi}{2} \\ -1; & \dfrac{\pi}{2} < x < \pi \end{cases}$ then $\hat{f}_C(p)$ is :

(a) $\dfrac{2}{p}\sin\left(\dfrac{\pi p}{2}\right)$ (b) $p\sin(\pi p)$

(c) $\dfrac{p}{2}\sin\left(\dfrac{\pi p}{2}\right)$ (d) None of these

20. If $f(x) = \sin nx$, n is positive integer and $p = n$ then $\hat{f}_S(p)$ is :

(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) 0

21. If $f(x) = 2x, 0 < x < 4$ then $\hat{f}_C(0)$ is :

(a) 0 (b) 1 (c) 4 (d) 16

22. If $n-p$ is odd then fourier finite cosine transform of $\sin nx$ is :

(a) $\dfrac{n}{n^2+p^2}$ (b) $\dfrac{2n}{n^2-p^2}$

(c) $\dfrac{2n}{n^2+p^2}$ (d) 0

23. The finite fourier sine transform of $\sin nx$ for $p \neq n$ is :

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{6}$ (d) 0

24. The finite fourier sine transform of $\left(1-\dfrac{x}{\pi}\right)$ is :

(a) 0 (b) p (c) $\dfrac{1}{p}$ (d) $\dfrac{1-p}{p}$

25. If $f(x) = \begin{cases} x; & 0 \le x < \dfrac{\pi}{2} \\ \pi - x; & \dfrac{\pi}{2} < x < \pi \end{cases}$ then $F_S\{f(x)\}$ is :

(a) $2\sin\dfrac{p\pi}{2}$ (b) $\dfrac{2}{p}\sin\dfrac{p\pi}{2}$

(c) $\dfrac{2}{p}\cos\dfrac{p\pi}{2}$ (d) $\dfrac{2}{p^2}\sin\dfrac{p\pi}{2}$

26. The finite fourier cosine transform of $\dfrac{x}{4\pi}$ is :

(a) $\dfrac{-1-(-1)^p}{4\pi p^2}$ (b) $\dfrac{-1+(-1)^p}{p^2}$

(c) $\dfrac{(-1)^p}{4\pi p^2}$ (d) None of these

27. If $p = n$ then finite fourier cosine transform of $\sin nx$ is :

(a) $\dfrac{2n}{n^2-p^2}$ (b) $\dfrac{n}{n^2-p^2}$

(c) np^2 (d) 0

28. If $p = 1, 2, 3, \ldots$ then finite fourier cosine transform of $3x^2$ is :

(a) $\dfrac{2\pi}{p}(-1)^p$ (b) $\dfrac{3\pi}{p^2}(-1)^p$

(c) $\dfrac{6\pi}{p^3}(-1)^p$ (d) $\dfrac{6\pi}{p^2}(-1)^p$

29. If $p = 0$ then finite fourier cosine transform of $\dfrac{x^2}{2\pi}-\dfrac{\pi}{6}$ is :

(a) 0 (b) $\dfrac{2\pi}{p}$ (c) $\dfrac{2\pi}{p^2}$ (d) $\dfrac{3\pi}{p^3}$

30. The finite fourier sine transform of $x(\pi - x)$ is :

(a) $\dfrac{1-(-1)^p}{p^2}$ (b) $\dfrac{2[1-(-1)^p]}{p^3}$

(c) $\dfrac{2[1+(-1)^p]}{p^3}$ (d) $\dfrac{1+(-1)^p}{p^3}$

31. The finite fourier cosine transform of $\left(\dfrac{\pi}{3}-x+\dfrac{x^2}{2\pi}\right)$ is :

(a) p (b) $\frac{1}{p}$ (c) $\frac{1}{p^2}$ (d) 0

32. If $p = 0$ then the finite fourier sine transform of $f(x) = a$ is :

(a) 0 (b) ap (c) $\frac{a}{p}$ (d) $a(-1)^p$

33. If $p = 0$ then the finite fourier cosine transform of $f(x) = Cx$ in the interval $0 < x < l$ is :

(a) Cpl (b) Cl (c) pl (d) $\frac{Cl^2}{2}$

34. The finite fourier cosine transform of Cx in $(0, l)$ is :

(a) $\frac{Cl}{\pi p}(\cos \pi p - 1)$ (b) $\frac{C^2 l^2}{\pi^2}(\cos \pi p - 1)$

(c) $\frac{\cos \pi p}{Cp^2}$ (d) $\frac{Cl^2}{\pi^2 p^2}(\cos \pi p - 1)$

35. If $p = 0$ then finite fourier cosine transform of $3x^2$ is :

(a) 0 (b) π (c) π^2 (d) π^3

36. If $p = 0$ then finite fourier cosine transform of $\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)$ is :

(a) p (b) $\frac{1}{p^2}$ (c) $\frac{2p}{\pi}$ (d) 0

37. If $p = 0$ then finite fourier cosine transform of $f(x) = a$ is :

(a) 0 (b) π (c) $a\pi$ (d) $\frac{\pi}{a}$

38. The finite fourier sine transform of Cx in $0 < x < l$ is :

(a) $\frac{l^2 C}{\pi p}(-1)^{p+1}$ (b) $-\frac{lC}{\pi p}$

(c) $(-1)^p \frac{\pi C}{pl^2}$ (d) None of these

39. If $f(x)$ is continuous and $f'(x)$ is sectionally continuous then for $p = 1, 2, \ldots$ $F_S\{f'(x)\}$ is :

(a) $-F_C\{f(x)\}$ (b) $-pF_C\{f(x)\}$

(c) $F_C\{f(x)\}$ (d) $pF_C\{f(x)\}$

40. If $F(x)$ and $G(x)$ be two functions on $-\pi < x < \pi$ then the convolution $F(x) * G(x)$ is :

(a) $\int_{-\pi}^{\pi} F(x)G(x)\,dx$ (b) $\int_{-\pi}^{\pi} F(x)G(y)\,dy$

(c) $\int_{-\pi}^{\pi} [F(x) - G(x)]\,dx$ (d) $\int_{-\pi}^{\pi} F(x)G(x - y)\,dy$

41. If $f(x)$ is continuous and $f'(x)$ is sectionally continuous then for $p = 0, 1, 2, \ldots$ $F_C\{f'(x)\}$ is :

(a) $pF_S\{f(x)\} - f(0) + (-1)^p f(\pi)$

(b) $pF_S\{f(x)\} - f(0)$

(c) $pF_S\{f(x)\} + (-1)^p f(\pi)$

(d) None of these

42. The finite fourier cosine transform of $f(x)$ on $(0, l)$ is :

(a) $\Sigma f(x) \cos \frac{\pi p x}{l}$ (b) $\int_0^l f(x) \cos px\,dx$

(c) $\int_0^l f(x) \cos \pi x\,dx$ (d) $\int_0^l f(x) \cos \frac{\pi p x}{l}\,dx$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(b)	**4.**	(d)	**5.**	(b)	**6.**	(d)	**7.**	(d)	**8.**	(d)	**9.**	(d)	**10.**	(d)
11.	(c)	**12.**	(d)	**13.**	(a)	**14.**	(d)	**15.**	(d)	**16.**	(b)	**17.**	(c)	**18.**	(a)	**19.**	(a)	**20.**	(d)
21.	(d)	**22.**	(b)	**23.**	(d)	**24.**	(c)	**25.**	(d)	**26.**	(a)	**27.**	(d)	**28.**	(d)	**29.**	(a)	**30.**	(b)
31.	(c)	**32.**	(a)	**33.**	(d)	**34.**	(d)	**35.**	(d)	**36.**	(d)	**37.**	(c)	**38.**	(a)	**39.**	(b)	**40.**	(d)
41.	(a)	**42.**	(d)																

HINTS AND SOLUTIONS

1. Given $f(x) = 1$

$$\hat{f}_S(p) = \int_0^{\pi} f(x) \sin px\,dx$$

$$= \int_0^{\pi} \sin px\,dx = -\left(\frac{\cos px}{p}\right)_0^{\pi}$$

$$= \frac{1 - \cos \pi p}{p} = \frac{1 - (-1)^p}{p}$$

2. $f(x) = x$,

so $\hat{f}_C(p) = \int_0^{\pi} f(x) \cos px\,dx = \int_0^{\pi} x \cos px\,dx$

$$= \left(\frac{x \sin px}{p}\right)_0^{\pi} - \frac{1}{p}\int_0^{\pi} \sin px\,dx = \left(\frac{\cos px}{p^2}\right)_0^{\pi}$$

$$= \frac{(-1) + (-1)^p}{p^2} \qquad i.e. \quad p = 1, 2, \ldots$$

3. $$\hat{f}_S(p) = \int_0^{\pi}\left(1-\frac{x}{\pi}\right)\sin px\,dx$$

$$= -\left[\left(1-\frac{x}{\pi}\right)\frac{\cos px}{p}\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{\pi}\right)\left(-\frac{\cos px}{p}\right)dx$$

$$= \frac{1}{p}-\frac{1}{\pi}\left(\frac{\sin px}{p^2}\right)_0^{\pi} = \frac{1}{p}$$

6. Given $f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x < \pi \end{cases}$

Then $f_C(0) = \int_0^{\pi} f(x)\cos px\,dx = \int_0^{\pi} f(x)\,dx$

$$= \int_0^{\pi/2} 1\,dx + \int_{\pi/2}^{\pi}(-1)\,dx = 0$$

7. When $p = 0$, $\cos px = 1$

$\therefore\ \hat{f}_C(p) = \int_0^{\pi} f(x)\cos px\,dx = \int_0^{\pi}\left(1-\frac{x}{\pi}\right)^2 dx$

$$= -\left[\frac{\pi}{3}\left(1-\frac{x}{\pi}\right)^3\right]_0^{\pi} = \frac{\pi}{3}$$

8. When $p = 0$, $\cos px = 1$, $f(x) = x^2$

So, $\hat{f}_C(p) = \int_0^{4} f(x)\cos\left(\frac{\pi px}{4}\right)dx = \int_0^{4} x^2 dx$

$$\hat{f}_C(p) = \frac{64}{3}$$

10. Given $f(x) = x$ and $p = 0$

$\hat{f}_C(p) = \int_0^{\pi} f(x)\cos px\,dx$

$$= \int_0^{\pi} x\,dx = \left(\frac{x^2}{2}\right)_0^{\pi} = \frac{\pi^2}{2}$$

11. $$\hat{f}_S(p) = \int_0^{\pi} f(x)\sin px\,dx$$

$$= \int_0^{\pi}\frac{x}{4\pi}\sin px\,dx$$

$$= \frac{1}{4\pi}\left[\left(-x\frac{\cos px}{p}\right)_0^{\pi} + \int_0^{\pi}\frac{\cos px}{p}dx\right]$$

$$= -\frac{1}{4p}\cos p\pi + \frac{1}{4\pi}\left[\frac{\sin px}{p^2}\right]_0^{\pi} = \frac{(-1)^{p+1}}{4p}$$

12. $$\because f(x) = \frac{1}{l}\hat{f}_C(0) + \frac{2}{l}\sum_{p=1}^{\infty}\hat{f}_C(p)\cos\left(\frac{\pi px}{l}\right)$$

$$= 1 + \frac{2}{1}\sum_{p=1}^{\infty}\frac{\cos\frac{2\pi p}{3}}{(2p+1)^2}\cos \pi px$$

$$= 1 + 2\,\Sigma\frac{\cos\left(2\pi\frac{p}{3}\right)}{(2p+1)^2}\cos \pi px$$

14. $$\hat{f}_S(p) = \int_0^{4} 2x\sin\left(\frac{p\pi x}{4}\right)dx = -\left(\frac{2x\cos\left(\frac{p\pi x}{4}\right)}{p\frac{\pi}{4}}\right)_0^{4}$$

$$+ 2\int_0^{4}\frac{4\cos\left(\frac{p\pi x}{4}\right)}{\frac{p\pi}{4}}dx$$

$$= -\frac{32}{p\pi}\cos p\pi + \frac{8}{p\pi}\left[\frac{\sin\left(\frac{p\pi x}{4}\right)}{\frac{p\pi}{4}}\right]_0^{4} = -\frac{32}{p\pi}\cos p\pi$$

16. Given $\hat{f}_S(p) = \frac{1-\cos \pi p}{p^2\pi^2}$, $0 < x < \pi$

$$\because f(x) = \frac{2}{\pi}\sum_{p=1}^{\alpha}\hat{f}_S(p)\sin px = \frac{2}{\pi}\sum_{p=1}^{\alpha}\left(\frac{1-\cos p\pi}{p^2\pi^2}\right)\sin px$$

$$= \frac{2}{\pi^3}\sum_{p=1}^{\alpha}\left(\frac{1-\cos p\pi}{p^2}\right)\sin px$$

18. $$\hat{f}_C(p) = \int_0^{\pi}\left(1-\frac{x}{\pi}\right)\cos px\,dx$$

$$= \left[\left(1-\frac{x}{\pi}\right)\frac{\sin px}{p}\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{\pi}\right)\frac{\sin px}{p}dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos p\pi}{p^2}\right]_0^{\pi} = \frac{1}{\pi p^2}[1-(-1)^p]$$

20. If $p = n$ and $f(x) = \sin nx$ then

$$\hat{f}_S(p) = \int_0^{\pi}\sin nx\sin nx\,dx = \int_0^{\pi}\sin^2 nx\,dx$$

$$= \int_0^{\pi}\left(\frac{1-\cos 2nx}{2}\right)dx = \frac{1}{2}\left[x-\frac{\sin 2nx}{2n}\right]_0^{\pi} = \frac{\pi}{2}$$

22. $$\hat{f}_C(p) = \int_0^{\pi}\sin nx\cos px\,dx$$

$$= \frac{1}{2}\int_0^{\pi}[\sin (n+p)\,x + \sin (n-p)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos (n+p)x}{n+p} - \frac{\cos(n-p)x}{n-p}\right]_0^{\pi} \text{ if } p \neq n$$

$$\therefore\ \hat{f}_C(p) = \frac{1}{2}\left[-\frac{\cos(n+p)\pi}{n+p} - \frac{\cos(n-p)\pi}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

If $n-p$ is odd then $n+p$ is also odd so

$$\hat{f}_C(p) = \frac{1}{2}\left[\frac{2}{n+p}+\frac{2}{n-p}\right] = \frac{2n}{n^2-p^2}$$

25. $\hat{f}_S(p) = \int_0^{\pi} f(x)\sin px\,dx$

$$= \int_0^{\pi/2} x\sin px\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin px\,dx$$

$$= \left[\frac{x(-\cos px)}{p} + \frac{\sin px}{p^2}\right]_0^{\pi/2}$$

$$+ \left[(\pi - x)\left(-\frac{\cos px}{p}\right) - \frac{\sin px}{p^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{p^2}\sin\left(\frac{\pi p}{2}\right)$$

29. If $p = 0$ then $\hat{f}_C(p) = \int_0^{\pi}\left(\frac{x^2}{2\pi} - \frac{\pi}{6}\right)dx$

$$= \left(\frac{x^3}{6\pi} - \frac{\pi}{6}x\right)_0^{\pi} = \frac{\pi^2}{6} - \frac{\pi^2}{6} = 0$$

31. $\hat{f}_C(p) = \int_0^{\pi}\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\cos px\,dx$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\frac{\sin px}{p}\right]_0^{\pi}$$

$$-\frac{1}{p}\int_0^{\pi}\left(-1 + \frac{x}{\pi}\right)\sin px\,dx$$

$$= -\frac{1}{p}\left[-\left(-1 + \frac{x}{\pi}\right)\frac{\cos px}{p}\right]_0^{\pi} - \frac{1}{p^2}\int_0^{\pi}\frac{\cos px}{\pi}dx$$

$$= \frac{1}{p^2} - \frac{1}{p^3\pi}(\sin px)_0^{\pi} = \frac{1}{p^2},\ p > 0$$

33. When $p = 0$, $\hat{f}_C(p) = \int_0^l Cx\cos 0x dx = \int_0^l Cx\,dx$

$$= \frac{Cl^2}{2}$$

35. $\hat{f}_C(0) = \int_0^{\pi} 3x^2\cos 0x\,dx = \int_0^{\pi} 3x^2 dx = \pi^3$

38. Here $f(x) = Cx,\ 0 < x < l$

$$\hat{f}_S(p) = \int_0^l f(x)\sin\frac{\pi px}{l}dx = \int_0^l Cx\sin\frac{\pi px}{l}dx$$

$$= C\left[-\left(xl\frac{\cos\frac{\pi px}{l}}{\pi p}\right)_0^l + \int_0^l l\frac{\cos\frac{pn\pi}{l}}{p\pi}dx\right] = \frac{l^2C}{\pi p}(-1)^{p+1}$$

39. $F_S\{f'(x)\} = \int_0^{\pi} f'(x)\sin px\,dx$

$$= \{f(x)\sin px\}_0^{\pi} - p\int_0^{\pi} f(x)\cos px\,dx$$

$$= -pF_C\{f(x)\},\ p = 1, 2, 3, \ldots$$

41. $F_C\{f'(x)\} = \int_0^{\pi} f'(x)\cos px\,dx$

$$= [f(x),\ \cos px]_0^{\pi} + p\int_0^{\pi} f(x)\sin px\,dx$$

$$= f(x)\cos p\pi - f(0) + pF_S\{f(x)\},\ p = 0,1\ 2, \ldots$$

$$= pF_S\{f(x)\} - f(0) + (-1)^p f(\pi)$$

* * *

CHAPTER 19

Applications of Fourier Transforms in Initial and Boundary Value Problems

APPLICATION OF INFINITE FOURIER TRANSFORMS

1. The choice of sine or cosine transform is decided by the form of boundary conditions at the lower limit of the variable selected for exclusion.

   For the exclusion of $\frac{\partial^2 V}{\partial x^2}$ from a differential equation we require

   (i) $(V)_{x=0}$ in sine transform.

   (ii) $\left(\frac{\partial V}{\partial x}\right)_{x=0}$ in cosine transform.

APPLICATION OF FINITE FOURIER TRANSFORMS

1. If the range of one of the variables is finite, finite Fourier transforms are applied.

   For the exclusion of $\frac{\partial^2 U}{\partial x^2}$ from a differential equation, we require

   (i) $U(0, t)$ and $U(l, t)$ in finite sine transform.

   (ii) $U_x(0, t)$ and $U_x(l, t)$ in finite cosine transform.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. For the exclusion of $\frac{\partial^2 V}{\partial x^2}$ from a differential equation if $(V)_{x=0}$ is given then we take

   (a) Fourier sine transform

   (b) Fourier cosine transform

   (c) Fourier transform

   (d) Laplace transform

2. In the solution of $\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}, U(0, t) = 0,$

   $U(x, 0) = e^{-x}, x > 0, U(x, t)$ is bonded for $x > 0, t > 0$ the new differential equation after Fourier sine transform is

   (a) $(D - p^2)\hat{U}_S = 0$ (b) $(D + 2p^2)\hat{U}_S = 0$

   (c) $(D - 2p^2)\hat{U}_S = 0$ (d) $(D + p^2)\hat{U}_S = 0$

3. For the exclusion of $\frac{\partial^2 V}{\partial x^2}$ from the differential equation we require

   (a) $\left(\frac{\partial V}{\partial x}\right)_{x=0}$ in sine transform

   (b) $(V)_{x=0}$ in cosine transform

   (c) $\left(\frac{\partial V}{\partial x}\right)_{x=0}$ in Laplace transform

   (d) $\left(\frac{\partial V}{\partial x}\right)_{x=0}$ in cosine transform

4. The finite fourier sine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $U(0, t) = U(u, t) = 0$, where $0 < x < 4, t > 0$ is given by

   (a) $\hat{U}_S = Ae^{-p^2\pi^2 t/16}$ (b) $\hat{U}_S = Ae^{p^2\pi^2 t}$

   (c) $\hat{U}_S = Ae^{-p^2\pi^2 t}$ (d) $\hat{U}_S = Ae^{p^2\pi^2 t/16}$

5. The finite fourier cosine transform of $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$ with $\frac{\partial V}{\partial x} = 0$ when $x = 0$ and $x = \pi, t > 0$ is :

   (a) $\hat{U}_C = Ae^{-pt}$ (b) $\hat{U}_C = Ae^{p^2 kt}$

   (c) $\hat{U}_C = Ae^{-p^2 kt}$ (d) $\hat{U}_C = Ae^{-p^2 t}$

6. The fourier transform of $\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2}$ for $x \geq 0$, $-\infty < b < \infty$ is :
(a) $\hat{U} = Ae^{px}$ (b) $\hat{U} = Ae^{-px}$
(c) $\hat{U} = Ae^{-p^2 x}$ (d) $\hat{U} = Ae^{p^2 x}$

7. The fourier finite cosine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ $0 < x < 6, t > 0, U_x(0, t) = U_x(6, t) = 0$ is :
(a) $\hat{U}_C = Ae^{-p^2\pi^2 t}$ (b) $\hat{U}_C = Ae^{-p^2\pi^2 t/16}$
(c) $\hat{U}_C = Ae^{-p^2\pi^2 t/32}$ (d) $\hat{U}_C = Ae^{-\frac{p^2\pi^2 t}{36}}$

8. The finite fourier sine transform of $\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^2 U}{\partial x^2}$, $0 < x < 2, t > 0, U(0, t) = U(2, t) = 0$ is :
(a) $\hat{U}_S = Ae^{-3p\pi \frac{t}{2}}$
(b) $\hat{U}_S = Ae^{-3\pi^2\pi^2 \frac{t}{2}}$
(c) $\hat{U}_S = A\cos\frac{3\pi t}{2} + B\sin\frac{3p\pi t}{2}$
(d) $\hat{U}_S A\cos h\frac{3p\pi t}{2} + B\sin h\frac{3p\pi t}{2}$

9. The finite fourier sine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ $0 < x < 6, t > 0$ and $U(0, t) = U(6, t) = 0$ is :
(a) $\hat{U}_S = Ae^{-p^2 \frac{t}{36}}$ (b) $\hat{U}_S = Ae^{-pt}$
(c) $\hat{U}_S = Ae^{-6pt}$ (d) $\hat{U}_S = Ae^{-p^2\pi^2 \frac{t}{36}}$

10. The finite fourier sine transform of $\frac{\partial^2 U}{\partial t^2} = 4\frac{\partial^2 U}{\partial x^2}$, $0 < x < \pi, t > 0$ and $U(0, t) = U(\pi, t) = 0$ is :
(a) $\hat{U}_S = Ae^{-2pt}$
(b) $\hat{U}_S = Ae^{4p^2 \frac{t^2}{\pi}}$
(c) $\hat{U}_S = A\cos 2pt + B\sin 2pt$
(d) $\hat{U}_S = A\cos h2pt + B\sin h2pt$

11. The finite fourier sine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0 < x < \pi, t > 0, U(0, t) = 1, U(\pi, t) = 3$ is :
(a) $(D + p^2)\hat{U}_S = p(1 + \cos \pi p)$
(b) $(D + p)\hat{U}_S = p(1 + 3\sin p\pi)$
(c) $(D + p)\hat{U}_S = p(1 - 3\sin p\pi)$
(d) $(D + p^2)\hat{U}_S = p(1 - 3\cos p\pi)$

12. The fourier transform of $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$, $-\alpha < x < \alpha$ is :
(a) $\hat{U} = Ae^{-kpt}$ (b) $\hat{U} = A\cos hpt$
(c) $\hat{U} = Ae^{-kp^2 t}$ (d) $\hat{U} = A\cos kp^2 t$

13. The fourier transform of $\frac{\partial^2 U}{\partial t^2} = C^2\frac{\partial^2 U}{\partial x^2}$ for $-\alpha < x < \alpha, t > 0$, is given by :
(a) $\hat{U} = Ae^{cpt}$
(b) $\hat{U} = A\cos cpt + B\sin cpt$
(c) $\hat{U} = A\cos hcpt + B\sin hcpt$
(d) $\hat{U} = Ae^{-cpt}$

14. The fourier sine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, x > 0, t > 0$ with $U(0, t) = 0$, $U = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$ when $t = 0$ and $U(x, t)$ is bounded, is :
(a) $\hat{U}_S = Ae^{-pt}$ (b) $\hat{U}_S = A\cos hpt$
(c) $\hat{U}_S = A\cos hp^2 t$ (d) $\hat{U}_S = Ae^{-p^2 t}$

15. Solve $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ for $\hat{U}_S$ when $U(0, y) = 0 = V(\pi, y)$:
(a) Ae^{-ipy}
(b) Be^{-ipy}
(c) $A\cos hpy + B\sin hpy$ (d) $Ae^{ipy} + Be^{-ipy}$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(a)	**5.**	(c)	**6.**	(c)	**7.**	(d)	**8.**	(c)	**9.**	(d)	**10.**	(c)
11.	(d)	**12.**	(c)	**13.**	(b)	**14.**	(d)	**15.**	(c)										

HINTS AND SOLUTIONS

2. Taking fourier sine transform of $\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial U}{\partial t}\sin px\,dx = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin px\,dx$$

$$\frac{d}{dt}\sqrt{\frac{2}{\pi}}\int_0^\infty U\sin px\,dx = 2\sqrt{\frac{2}{\pi}}\left[\left(\frac{\partial U}{\partial x}\sin px\right)_0^\infty - p\int_0^\infty \frac{\partial U}{\partial x}\cos px\,dx\right]$$

if $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$ so

$$\frac{d\hat{U}_S}{dt} = -2p\sqrt{\frac{2}{\pi}}\,(U\cos px)_0^\alpha - 2p^2\sqrt{\frac{2}{\pi}}\int_0^\alpha U\sin px\,dx$$

if $U \to 0$ as $x = \infty$ we get

$$= t2p\sqrt{\frac{2}{\pi}}\,U(0, t) - 2p^2\hat{U}_S$$

or $$\frac{d\hat{U}_S}{dt} + 2p^2\hat{U}_S = 2p\sqrt{\frac{2}{\pi}}\,U(0, t) = 0$$

$\therefore$ $$\hat{U}_S = Ae^{-2p^2t}$$

5. Taking finite fourier cosine transform of

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$$

$$\int_0^\pi \frac{\partial U}{\partial t}\cos px\,dx = k\int_0^\pi \frac{\partial^2 U}{\partial x^2}\cos px\,dx$$

$$\frac{d\hat{U}_C}{dt} = k[-p^2\hat{U}_C - \{U_x(0, t) - U_x(\pi, t)\cos\pi p\}]$$

or $$\frac{dU_C}{dt} = -kk^2\hat{U}_C$$

Integrates it $\hat{U}_C = Ae^{-kp^2t}$

6. Taking the fourier transform of both sides of

$$\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} \text{ we have}$$

$$\int_{-\infty}^\infty \frac{\partial U}{\partial x}e^{ipy}dy = \int_{-\infty}^\infty \frac{\partial^2 U}{\partial y^2}e^{ipy}dy$$

or $$\frac{d}{dx}\int_{-\infty}^\infty Ue^{ipy}dy = (-ip)^2\hat{U}$$

$$\frac{d}{dx}\hat{U} = -p^2\hat{U} \quad\Rightarrow\quad \hat{U} = Ae^{-p^2x}$$

7. Taking finite fourier cosine transform of

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \text{ we have}$$

$$\int_0^6 \frac{\partial U}{\partial t}\cos\frac{\pi px}{6}dx = \int_0^6 \frac{\partial^2 U}{\partial x^2}\cos\frac{\pi px}{6}dx$$

$$\frac{d\hat{U}_C}{dt} = \frac{-p^2\pi^2}{6^2}\hat{U}_C - \{U_x(0, t) - U_x(6, t)\cos p\pi\}$$

or $$\frac{d\hat{U}_C}{dt} = \frac{-\pi^2p^2}{36}\hat{U}_C$$

So $$\hat{U}_C = Ae^{-\left(\frac{p^2\pi^2t}{36}\right)}$$

8. Taking finite fourier sine transform of $\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^2 U}{\partial x^2}$,

we have

$$\int_0^2 \frac{\partial^2 U}{\partial t^2}\sin\frac{p\pi x}{2}dx = a\int_0^2 \frac{\partial^2 U}{\partial x^2}\sin\frac{p\pi x}{2}dx$$

or $$\frac{\partial^2\hat{U}_S}{dt^2} = -\frac{9p^2\pi^2}{4}\hat{U}_S + \frac{9p\pi}{4}[U(0, t) - U(2, t)\cos p\pi]$$

$$\frac{d^2\hat{U}_S}{dt^2} + \frac{9p^2\pi^2}{4}\hat{U}_S = 0$$

So, the solution is $\hat{U}_S = A\cos\frac{3p\pi t}{2} + B\sin\frac{3p\pi t}{2}$.

10. Taking Laplace transform of $\frac{\partial^2 U}{\partial t^2} = 4\frac{\partial^2 U}{\partial x^2}$

$$\int_0^\pi \frac{\partial^2 U}{\partial t^2}\sin px\,dx = 4\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin px\,dx$$

$$\frac{d^2\hat{U}_S}{dt^2} = -4p^2\hat{U}_S + 4p\,\{U(0, t) - U(\pi, t)\cos p\pi\}$$

$\because$ $U(0, t) = U(\pi, t) = 0$, we get

$$\frac{d^2\hat{U}_S}{dt^2} + 4p^2\hat{U}_S = 0$$

$\Rightarrow$ $(D^2 + 4p^2)\,\hat{U}_S = 0$

So the required solution is

$$\hat{U}_S = A\cos 2pt + B\sin 2pt$$

11. Taking the finite fourier sine transform of

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

$$\int_0^\pi \frac{\partial U}{\partial t}\sin px\,dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin px\,dx$$

$$\frac{d\hat{U}_S}{dt} - p^2\hat{U}_S + p\,\{U(0, t) - U(\pi, t)\cos p\pi\}$$

$\because$ $U(0, t) = 1, U(\pi, t) = 3$, we get

$$\frac{d\hat{U}_S}{dt} = p^2\hat{U}_S + p\,(1-3\cos p\pi)$$

or
$$\frac{d\hat{U}_S}{dt} + p^2\hat{U}_S = p\,(1-3\cos p\pi)$$

12. Taking the fourier transform of $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial U}{\partial t}e^{ipx}dx = \frac{k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 U}{\partial x^2}e^{ipx}dx$$

$$\frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Ue^{ipx}dx = k\,(-ip)^2\hat{U}$$

$$\frac{d\hat{U}}{dt} = -kp^2\hat{U}$$

So
$$\hat{U} = Ae^{-kp^2t}$$

13. Taking the fourier transform of $\frac{\partial^2 U}{\partial t^2} = c^2\frac{\partial^2 U}{\partial x^2}$, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 U}{\partial t^2}e^{ipx}dx = \frac{c^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 U}{\partial x^2}e^{ipx}dx$$

$$\frac{d^2}{dt^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Ue^{ipx}dx = c^2(-ip)^2\hat{U}$$

$$\frac{d^2\hat{U}}{dt^2} = -c^2p^2\hat{U}$$

or
$$\frac{d^2\hat{U}}{dt^2} + c^2p^2\hat{U} = 0$$

A.E. is $m^2 + c^2p^2 = 0$

$$m = \pm\, icp$$

where solution is $\hat{U} = A\cos cpt + B\sin cpt$.

14. Since $(U)_{x=0}$ is given so taking the fourier sine transform of $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial U}{\partial t}\sin px\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial^2 U}{\partial x^2}\sin px\,dx$$

$$\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty U\sin px\,dx = \sqrt{\frac{2}{\pi}}\left\{\frac{\partial U}{\partial x}\sin px\right\}_0^\infty - p\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial U}{\partial x}\cos px\,dx$$

$$\frac{d\hat{U}_S}{dt} = p\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial U}{\partial x}\cos px\,dx$$

if $\frac{\partial U}{\partial x}\to 0$ as $x\to\infty$

$$= -p\sqrt{\frac{2}{\pi}}[U(x,t)\cos px]_0^\infty - p^2\sqrt{\frac{2}{\pi}}\int_0^\infty U\sin px\,dx$$

$$= \sqrt{\frac{2}{\pi}}pU\,(0,t) - p^2\hat{U}_S \text{ as } U\to 0 \text{ as } x\to\infty$$

$$\frac{d\hat{U}_S}{dt} + p^2\hat{U}_S = \sqrt{\frac{2}{\pi}}pU(0,t)$$

$$\frac{d\hat{U}_S}{dt} + p^2\hat{U}_S = 0 \quad \because\ U(0,t) = 0$$

So, its solution is $\hat{U}_S = Ae^{-p^2t}$.

15. Taking finite fourier sine transform of $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$

$$\int_0^\pi\frac{\partial^2 U}{\partial x^2}\sin px\,dx + \int_0^\pi\frac{\partial^2 U}{\partial y^2}\sin px\,dx = 0$$

$$\left[\frac{\partial U}{\partial x}\sin px\right]_0^\pi - p\int_0^\pi\frac{\partial U}{\partial x}\cos px\,dx + \frac{\partial^2}{\partial y^2}\int_0^\pi U\sin px\,dx = 0$$

or $-p[U\cos px]_0^\pi - p^2\int_0^\pi U\sin px\,dx + \frac{d^2\hat{U}_S}{dy^2} = 0$

or
$$\frac{d^2U_S}{dy^2} - p^2\hat{U}_S = 0$$

Since $U(0,y) = U\,(\pi,y) = 0$

So, the solution is $\hat{U}_S = A\cos hpy + B\sin hpy$.

* * *

Unit-V

CHAPTER

20

Fourier Series

FOURIER SERIES EXPANSION IN THE INTERVAL $(0, 2\pi)$

1. The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} bn \sin nx$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

2. If the Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} bn \sin nx$$

Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

FOURIER SERIES EXPANSION IN $(-\pi, \pi)$

1. The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

1. Let $f(x) = \begin{cases} f_1(x), & \alpha < x < x_0 \\ f_2(x), & x_0 < x < \alpha + 2\pi \end{cases}$

in the interval $(\alpha, \alpha + 2\pi)$ and x_0 is the point of discontinuity. Here the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} bn \sin nx$$

Where $a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)dx\right]$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cos nx dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\sin nx\, dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\sin nx\, dx\right]$$

At the point of discontinuity, $x = \alpha$ the Fourier series gives $f(x)$ as

$$f(x) = \frac{1}{2}[f(\alpha - 0) + f(\alpha + 0)]$$

CHANGE OF PERIOD

Let the function $f(x)$ in $(0, 2c)$

$\because$ $2c$ is the interval for the variable $= x$

$\therefore$ 2π is the interval for the variable $= \frac{x}{2c}.\,2\pi$

$$= \frac{\pi x}{c}$$

So, putting $z = \frac{\pi x}{c}$ or $x = \frac{zc}{\pi}$, the function $f(x)$ of period $2c$ is transformed for the function $f\left(\frac{cz}{\pi}\right)$ of period 2π.

FOURIER SERIES EXPANSION OF EVEN OR ODD FUNCTION IN $(-\pi, \pi)$

1. Let the fourier series expansion of an even function in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

2. Let the Fourier series expansion of odd function $[f(-x) = -f(x)]$ is

$$f(x) = \frac{a_0}{2} + \int_{n=1}^{\infty} a_n \sin x + \int_{n=1}^{\infty} n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0,$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

HALF RANGE FOURIER COSINE AND SINE SERIES IN (0, T)

1. The half range Fourier cosine series in $(0, T)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$$

where $a_0 = \frac{2}{T}\int_{0}^{T} f(x)\,dx,$

$$a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}\,dx$$

2. The half range Fourier sine series in $(0, T)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T}$$

where $b_n = \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}\,dx$

PARSEVAL'S FORMULA

1. If the Fourier series for $f(x)$ converges uniformly in $(-l, l)$ then

$$\int_{-l}^{l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

2. If the Fourier series for $f(x)$ converges uniformly in $(0, 2l)$ then

$$\int_{0}^{2l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. If the Fourier series of $f(x)$ has only cosine forms then $f(x)$ must be :

(a) Even (b) Odd

(c) May be even or odd (d) Does not exist

2. The coefficient a_0 in the Fourier series for e^{-x} in $0 < x < 2\pi$ is :

(a) $\frac{(1-e^{-2\pi})}{n^2+1}$ (b) $(1-e^{-2\pi})\frac{n}{n^2+1}$

(c) $\frac{1-e^{-2\pi}}{2\pi}$ (d) $\frac{(1-e^{-2\pi})}{2\pi n}$

3. The coefficient a_n in the Fourier series for $x \cos x$ in $-\pi < x < \pi$ is :

(a) 0 (b) $\frac{2n}{n^2-1}$

(c) $\frac{-n}{n^2-1}$ (d) None of thse

4. The coefficient b_n in the Fourier series expansion of x^3 for $-\pi < x < \pi$ is :

(a) 0 (b) $(-1)^n\left[\frac{\pi}{n^3} - \frac{6}{n}\right]$

(c) $(-1)^n\frac{\pi}{n^3}$ (d) $2(-1)^n\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right]$

5. The coefficient a_0 in the expansion of $f(x) = e^x$ in $(0, 1)$ is :

(a) $2e$ (b) $2e-1$ (c) $2e-2$ (d) $2e-3$

6. The coefficeint b_n in the expansion of $f(x) = 1$ in $r < x < \pi$ is :

(a) $\begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ (b) $\begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

(c) $\frac{4}{n\pi}$ (d) $\frac{n\pi}{4}$

7. At the point of discontinuity α, Fourier series converges :

(a) $f(\alpha+0)+f(\alpha-0)$ (b) $f'(\alpha+0)+f'(\alpha-0)$

(c) $\frac{1}{2}[f(\alpha+0)+f(\alpha-0)]$ (d) $\frac{f(\alpha+0)-f(\alpha-0)}{2}$

8. Fourier coefficient a_0 in Fourier series expansion represents :

(a) Minimum value of $f(x)$

(b) Maximum value of $f(x)$

(c) Mean value of $f(x)$

(d) None of these

9. In the Fourier series expansion of $f(x)=|x|$ in $(-\pi, \pi)$ the value of a_2 is :

(a) 0 (b) $-\frac{4}{9\pi}$ (c) $\frac{3}{4\pi}$ (d) $-\frac{2}{5\pi}$

10. $\cos \pi x$ definied in $(0, 1)$ can be expanded in series of :

(a) Cosine terms only

(b) Sine and cosine terms both

(c) Sine terms only

(d) Constant terms only

11. In the Fourier series expansion of $f(x)=x-x^2$ in $(-\pi, \pi)$ the value of a_n is :

(a) $\frac{(-1)^n}{n^2}$ (b) $\frac{(-1)^{n+1}}{n^2}$

(c) $\frac{-n(-1)^n}{n^2}$ (d) $\frac{2(-1)^{n+1}}{n^2}$

12. If $f(x)=\begin{cases}-\pi & -\pi<x<0\\ x & 0<x<\pi\end{cases}$ then the coefficient a_0 in the Fourier series expansion is :

(a) $(-1)^n\frac{\pi}{2}$ (b) $\frac{\sin n\pi}{\pi}$

(c) $\frac{-\pi^2}{2}$ (d) $\frac{-\pi}{2}$

13. The coefficient a_0 in the fourier series for x^2 in $-\pi<x<\pi$ is :

(a) $n\pi$ (b) $2n\pi^2$ (c) $\frac{n\pi^2}{3}$ (d) $\frac{\pi^3}{3}$

14. In the expansion of $1-\frac{x}{e}$ in $(0, l)$ the coefficient a_0 is :

(a) $\frac{4}{n^2\pi^2}$ (b) $\frac{2}{n\pi}$ (c) $2n^2\pi^2$ (d) 1

15. The coefficient a_0 in the Fourier series expansion of $\sin\left(\frac{\pi x}{l}\right)$ in $0<x<l$ is :

(a) $\frac{\pi}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{4}{\pi^2}$ (d) $\frac{4}{\pi}$

16. The Fourier series of $\sin 3x$ in $(-\pi, \pi)$ is :

(a) $\sin x-\frac{\sin 3x}{3}$ (b) $x-\frac{x^3}{3!}+\frac{x^5}{5!}$

(c) $\frac{3}{4}\sin x-\frac{\sin 3x}{4}$ (d) $3\sin x-\frac{\sin 3x}{3}$

17. In the Fourier series expansion of x^2nx $(-\pi, \pi)$ the coefficient a_n is :

(a) $\frac{4}{n^2}(-1)^n$ (b) $\frac{(-1)^n}{n^2}$

(c) $\frac{(-1)^n}{n}$ (d) $\frac{4(-1)^n}{n}$

18. If $f(x)=\begin{cases}x+\pi & r\le x\le\pi\\ -x-\pi & -\pi\le x<0\end{cases}$ and $f(x)$ is of period 2π then the coefficient a_0 is :

(a) 0 (b) 1 (c) π^2 (d) π

19. The coefficient a_n in the Fourier expansion of $\frac{l}{2}-x$ in $(0, l)$ is :

(a) $\frac{l}{n\pi}$ (b) $\frac{n\pi}{l}$ (c) $\frac{n}{l\pi}$ (d) 0

20. If $f(x)=\begin{cases}x+1 & -1<x<0\\ x-1 & r<x<1\end{cases}$, then coefficient b_n in the Fourier series is :

(a) $\frac{1}{n\pi}$ (b) $\frac{-2}{n\pi}$ (c) $\frac{n\pi}{2}$ (d) 0

21. The coefficient a_0 in the Fourier expansion of $f(x)=\begin{cases}\frac{1}{2}+x & -\frac{1}{2}<x\le 0\\ \frac{1}{2}-x & 0<x<\frac{1}{2}\end{cases}$ is :

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) 0

22. If $f(x)=\begin{cases}0, & 0<x<C\\ 1, & C<x<2C\end{cases}$ then the coefficent a_n in the Fourier series is :

(a) $\frac{n\pi}{2}$ (b) $-\frac{2}{n\pi}$ (c) $-\frac{1}{n\pi}$ (d) 0

23. The coefficient a_n in the Fourier expansion of x^2 in $(-\pi, \pi)$ is :

(a) $\frac{4(-1)^n}{n^2}$ (b) $\frac{(-1)^n}{n^2}$ (c) $(-1)^n n^2$ (d) 0

24. The Fourier series of $\cos 3x$ in $(-\pi, \pi)$ is :

(a) $\frac{3}{4}\cos x+\frac{\cos 3x}{4}$ (b) $\cos x+\frac{\cos 3x}{3!}+\ldots$

(c) $\cos x+\frac{3}{4}\cos 3x$ (d) $1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$

25. In the Fourier expansion of $x - x^2$ in $(-\pi, \pi)$ the coefficient b_2 is :

(a) 1 (b) –2 (c) –1 (d) –4

26. The coefficient b_n in the Fourier series expansion of $f(x) = x^2$ in $(-\pi, \pi)$ is :

(a) $\frac{(-1)^n}{n}$ (b) $\frac{n(-1)^4}{n^2}$

(c) $2n^2$ (d) 0

27. If $f(x) = \begin{cases} -1 & -\pi < x < -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$, then Fourier coefficient a_0 is :

(a) $\cos\frac{n\pi}{2} - \pi$ (b) $\sin\frac{n\pi}{2} - \pi$

(c) $\sin\frac{n\pi}{2}$ (d) 0

28. The Fourier coefficient b_n in the Fourier expansion of $\frac{l}{2} - x$ in $(0, l)$ is :

(a) $\frac{l}{n\pi}$ (b) $\frac{n\pi}{l}$ (c) $\frac{\pi}{ln}$ (d) 0

29. If $f(x) = \begin{cases} 1 + \frac{2x}{l} & l < x < 0 \\ 1 - \frac{2x}{l} & 0 < x < l \end{cases}$, then coefficient a_0 is :

(a) 0 (b) $\frac{8}{n^2\pi^2}$ (c) $\frac{n^2\pi^2}{8}$ (d) $\frac{n\pi}{2}$

30. The coefficient a_n in the Fourier expansion of $f(x) = \begin{cases} x+1 & -1 < x < 0 \\ x-1 & 0 < x < 1 \end{cases}$ is:

(a) $n\pi$ (b) $\frac{n^2\pi}{2}$ (c) $\frac{-2}{n\pi}$ (d) 0

31. The coefficient b_n in the Fourier expansion of $f(x) = \begin{cases} \frac{1}{2} + x & -\frac{1}{2} < x \le 0 \\ \frac{1}{2} - x & 0 < x < \frac{1}{2} \end{cases}$ is :

(a) $\frac{2}{n^2\pi^2}$ (b) $\frac{n\pi}{2}$

(c) $n^2\pi^2$ (d) 0

32. If $f(x) = \begin{cases} 0 & 0 < x < C \\ 1 & C < x < 2C \end{cases}$, then the coefficient b_n in the Fourier expansion is :

(a) $\frac{2}{n\pi}$ (b) $-\frac{2}{n\pi}$ if n is odd otherwise 0

(c) $\frac{2}{n\pi}$ if n is even otherwise 0

(d) $-\frac{2}{n\pi}$

33. The coefficient a_0 in the Fourier expansion of $|x|$ in $(-2, 2)$ is :

(a) 1 (b) 2 (c) 3 (d) 0

34. If $f(x) = x(\pi - x)$ then the coefficient a_0 in the Fourier expansion in the interval $(0, \pi)$ is :

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi^2}{2}$

35. The coefficient a_2 in the Fourier expansion of $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$ is :

(a) $\frac{1}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{-2}{\pi}$ (d) $\frac{-4}{\pi}$

36. The Fourier series expansion in complex form is :

(a) $\sum_{n=-\infty}^{\infty} C_n e^{-in\pi\frac{x}{t}}$ (b) $\sum_{n=-\infty}^{\infty} C_n e^{n\pi x/T}$

(c) $\sum_{n=-\infty}^{\infty} C_n e^{-n\pi x/T}$ (d) $\sum_{n=-\infty}^{\infty} C_n e^{+tin\pi x/T}$

37. The coefficient a_0 in the Fourier expansion of $f(x) \begin{cases} \pi x, & r < x < 1 \\ \pi\,(2-x) & 1 < x < 2 \end{cases}$ is :

(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^3}{3}$ (c) π (d) 0

38. If $f(x) = x(\pi - x)$ then the coefficient b_n in $(0, \pi)$ is :

(a) $\frac{\pi n^3}{8}$

(b) $\frac{8}{n^3\pi}$ if n is odd otherwise 0

(c) $\frac{n^3\pi^3}{8}$

(d) $\frac{8}{n^3\pi}$ if n is even otherwise 0

39. The coefficient b_n in the Fourier expansion of $|x|$ in $(-2, 2)$ is :

(a) $\frac{8}{n^2\pi^2}$ (b) $\frac{(-1)^n}{n^2\pi^2}$ (c) $n^2\pi^2$ (d) 0

40. If $f(x) = x^4$ in $(-1, 1)$ then the Fourier coefficient b_n is :

(a) 1 (b) 2 (c) 3 (d) 0

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(a)	**2.**	(c)	**3.**	(a)	**4.**	(d)	**5.**	(c)	**6.**	(a)	**7.**	(c)	**8.**	(c)	**9.**	(a)	**10.**	(c)
11.	(c)	**12.**	(d)	**13.**	(d)	**14.**	(d)	**15.**	(d)	**16.**	(c)	**17.**	(a)	**18.**	(d)	**19.**	(d)	**20.**	(b)
21.	(b)	**22.**	(d)	**23.**	(a)	**24.**	(a)	**25.**	(c)	**26.**	(d)	**27.**	(d)	**28.**	(a)	**29.**	(a)	**30.**	(d)
31.	(d)	**32.**	(b)	**33.**	(b)	**34.**	(b)	**35.**	(d)	**36.**	(d)	**37.**	(c)	**38.**	(b)	**39.**	(d)	**40.**	(d)

HINTS AND SOLUTIONS

2. $f(x) = e^{-x}$ in $(0, 2\pi)$

So, $a_0 = \dfrac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \dfrac{1}{2\pi}\int_0^{2\pi} e^{-x}\,dx = \dfrac{1-e^{-2\pi}}{2\pi}$

4. $f(x) = x^3$ for $(-\pi, \pi)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin px\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin px\,dx$$

$$\frac{2}{\pi}\left[\frac{-\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3} = 2(-1)^n\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)\right]$$

6. $f(x) = 1,\ r < x < \pi$

So, $b_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \dfrac{2}{\pi}\int_0^{\pi}\sin nx\,dx$

$$\frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = -\frac{2}{n\pi}[(-1)^n - 1]$$

$$= \begin{cases} \dfrac{n}{4\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

9. $f(x) = |x|$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} -x\cos nx\,dx$$

$$= -\frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{+1} = \frac{2}{\pi n^2}[1-(-1)^n]$$

$= \dfrac{4}{\pi n^2}$ if n is odd and 0 if n is even.

So, $a_2 = 0$

11. $f(x) = x - x^2$ in $(-\pi, \pi)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2}$$

12. $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ \pi & r < x < \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{+1} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left(\pi^2 + \frac{\pi^2}{2}\right)\right] = -\frac{\pi}{2}$$

14. $f(x) = 1 - \dfrac{x}{l},\ 0 \le x \le l$

Then $a_0 = \dfrac{2}{l}\int_0^{l} f(x)\,dx = \dfrac{2}{l}\int_0^{l}\left(1 - \dfrac{x}{l}\right)dx$

$$= \frac{2}{l}\left[-\frac{x^2}{2l} + x\right]_0^{l} = \frac{2}{l}\left[-\frac{l^2}{2l} + l\right]$$

$$= \frac{2l}{l}\left[1 - \frac{1}{l}\right] = 1$$

16. $\because \sin 3x = 3\sin x - 4\sin^3 x$

so, $\sin^3 x = \dfrac{3}{4}\sin x - \dfrac{\sin 3x}{4}$

18. $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\,dx + \dfrac{1}{\pi}\int_0^{\pi}(x+\pi)\,dx$

$$= \frac{1}{\pi}\left\{\left(\frac{-x^2}{2} - \pi x\right)_{\pi}^{0} + \left(\frac{x^2}{2} + \pi x\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \pi^2\right) + \left(\frac{\pi^2}{2} + \pi^2\right)\right] = \pi$$

20. $b_n = \dfrac{1}{1}\int_{-1}^{1} f(x)\sin\dfrac{n\pi x}{t}\,dx$

$$= \int_{-1}^{0}(x+1)\sin nx\pi\,dx + \int_0^{1}(x-1)\sin n\pi x\,dx$$

$$= \left[(x+1)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0}$$

$$+ \left[(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1} = -\frac{2}{n\pi}$$

22. $$a_n = \frac{1}{C}\int_0^{2C} f(x)\cos\frac{n\pi x}{C}\,dx$$

$$= \frac{1}{C}\int_0^{C} 0.\cos\frac{n\pi x}{C}\,dx + \frac{1}{C}\int_C^{2C} 1.\cos\frac{n\pi x}{C}\,dx$$

$$= \frac{1}{C}\left[\frac{C}{4\pi}\sin\frac{4\pi x}{C}\right]_0^{2\pi} = 0$$

25. $f(x) = x - x^2$ in $(-\pi, \pi)$

$$\because\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right)\right.$$

$$\left.-(1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-2)\,(-1)^n}{n}$$

$$\therefore b_2 = \frac{(-2)\,(-1)^2}{2} = -1$$

27. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi/2}(-1)dx\right.$$

$$\left.+\int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} 1\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = 0$$

29 $$a_0 = \frac{1}{2C}\int_{-C}^{C} F(x)\,dx = \frac{1}{2l}\int_{-l}^{l} F(x)\,dx$$

$$= \frac{1}{2l}\left[\int_{-l}^{0}\left(1+\frac{2x}{l}\right)dx + \int_0^{l}\left(1-\frac{2x}{l}\right)dx\right]$$

$$= \frac{1}{2l}\left[x + \frac{x^2}{l}\right]_{-l}^{0} + \frac{1}{2l}\left[x - \frac{x^2}{l}\right]_0^{l} = 0$$

31. $$b_n = \frac{1}{\frac{1}{2}}\int_{-1/2}^{1/2} f(x)\sin\frac{n\pi x}{\frac{1}{2}}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\ dx$$

$$+ 2\int_0^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[-\frac{1}{4n\pi}\right] + 2\left[\frac{1}{4n\pi}\right] = 0$$

33. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_0^{2} x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

34. $$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

35. $$a_2 = \frac{2}{C}\int_0^{C} f(x)\cos\frac{4\pi x}{C}\,dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \pi\left[\frac{x\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 +$$

$$\pi\left[(2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi x\right]$$

$$a_n = \frac{4}{n^2\pi}\left[2\cos\frac{4\pi}{2} - 1 - \cos n\pi\right]$$

$$\therefore\quad a_2 = -\frac{4}{\pi}$$

37. $$a_0 = \frac{2}{C}\int_0^{C} f(x)\,dx = \frac{2}{2}\left[\int_0^1 \pi x dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi$$

38. $$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right)\right.$$

$$\left.+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\frac{(-1)^4}{n^3} + \frac{2}{n^3}\right] = \begin{cases}\dfrac{8}{n^3\pi} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

40. $$b_n\int_{-1}^{1} f(x)\sin nx dx = \int_{-1}^{1} x^4\sin nx\,dx = 0$$

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